UNIT 07

Kinematics

Simple notes with all steps, graphs and the solved Exercise 7 (Q1 to Q13)

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Introduction

- **Kinematics** is the branch of mechanics that describes the motion of objects. It does **not** ask what causes the motion (forces).
- In this unit we study motion along a straight line. The main quantities are **displacement**, **velocity** and **acceleration**.
- We sketch and read **displacement-time** and **velocity-time** graphs. Graphs give a clear picture of the motion.
- We use **differentiation** and **integration** with respect to time to see how displacement, velocity and acceleration change.
- For constant acceleration we use the **equations of motion** to find unknown quantities.
- Finally we apply the ideas to real life: motion of vehicles, free fall, vertical projectile motion, relative motion, and realistic movement in animation.

Content outline

- Rectilinear motion
- Displacement, average velocity and acceleration
- Displacement-time graph
- Velocity-time graph
- Motion with constant acceleration
- Motion of a free particle moving along a vertical line

1. Rectilinear motion

The motion of a particle along a **straight line** is called **rectilinear motion**. The quantities used to describe motion are of two kinds.

- **Scalars** have magnitude only (size or number). They have no direction.
- **Vectors** have both magnitude and direction.

Quantity	Type	Meaning
Distance	Scalar	Total length of the path travelled, whatever the direction
Speed	Scalar	Distance covered per unit time, whatever the direction
Displacement	Vector	Change in position, from the starting point to the final point
Velocity	Vector	Rate of change of displacement with time, in a given direction. Its size is the speed
Acceleration	Vector	Rate of change of velocity with time, including direction

Displacement, average velocity and average speed

In rectilinear motion, displacement, velocity and acceleration can point in only **one of two directions**: along the line of motion, one way or the other. So we do not need arrows or vector symbols. We use **plus and minus signs** instead.

- One direction along the line is taken as **positive**.
- The opposite direction is taken as **negative**.

This makes writing displacement, velocity and acceleration simple.

Average velocity is the total displacement divided by the total time taken.

$$v_{avg} = \frac{\text{Total displacement}}{\text{Total time}}$$

Average speed is the total distance travelled divided by the total time taken.

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time}}$$

> **Recall**
>
> A body goes out and comes back to its starting position. Then:
>
> • Displacement = 0 (the start and end points are the same).
>
> • Distance ≠ 0 (the body still travelled a path).

2. Graphs of motion

Displacement-time graph

Time is on the x-axis and displacement is on the y-axis. The **slope** of this graph gives the **velocity**, and its sign gives the direction.

> **Recall**
>
> Positive slope → forward motion.
>
> Negative slope → backward motion.

Velocity-time graph: key features

Time (t) is on the x-axis and velocity (v) is on the y-axis.

Feature 1: the slope of the graph is the acceleration.

$$\text{Slope} = \frac{dv}{dt} = \text{acceleration } (a)$$

- If the acceleration is **constant**, the velocity-time graph is a **straight line**.
- If a = dv/dt = 0, the graph is a **horizontal straight line**. The particle moves with uniform velocity. In Figure 7.4 this is from point B to point C.
- If a = dv/dt = constant > 0, the velocity **increases** with time and the graph is a **rising** straight line. In Figure 7.4 the particle moves with positive uniform acceleration from A to B.
- If a = dv/dt = constant < 0, the velocity **decreases** with time and the graph is a **falling** straight line. In Figure 7.4 the particle moves with uniform negative acceleration (retardation) from C to D.
- If the acceleration is **variable**, the graph is **not a straight line**. It is a curve (Figure 7.5).

Figure 7.4 Constant acceleration

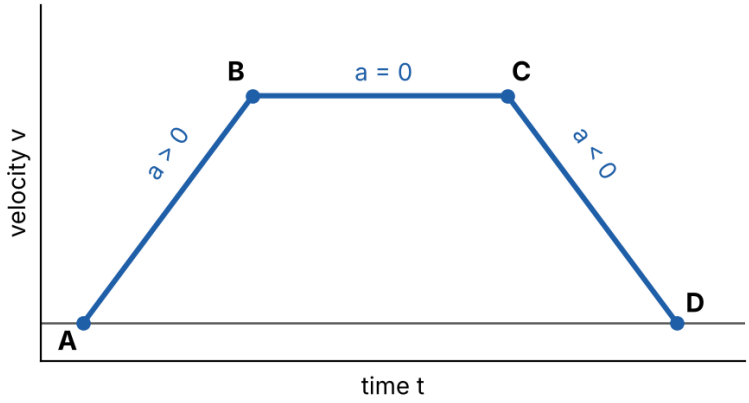
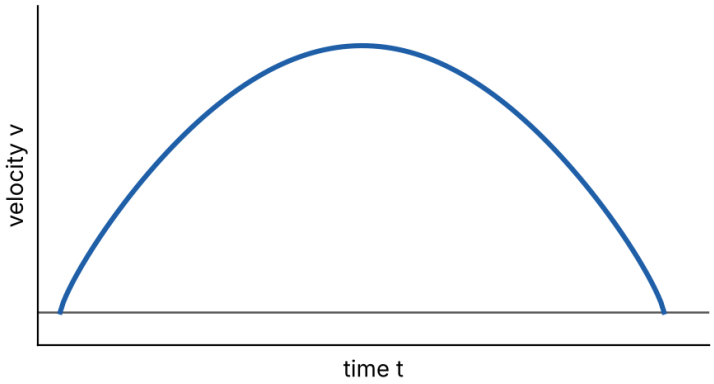


Figure 7.5 Variable acceleration (curved graph)



> **Recall**
>
> Displacement = area under the velocity-time graph.
>
> Acceleration = slope of the velocity-time graph.

Feature 2: the area under the graph is the displacement.

Let A be the area under the velocity-time graph, above the time axis, from t = t_1 to t = t_2.

> **Recall**
>
> If y = f(x) ≥ 0 for a ≤ x ≤ b, the area under the curve, above the x-axis and between the lines x = a and x = b, is the integral of y dx from a to b.

Using this recall with v in place of y and t in place of x, the area is an integral of the velocity.

$$A = \int_{t_1}^{t_2} v(t)\,dt$$

Velocity is the rate of change of displacement, so v = dx/dt. Put this in. The dt cancels, and the integral of dx is x.

$$A = \int_{t_1}^{t_2} \frac{dx}{dt}\,dt = \int_{t_1}^{t_2} dx = x(t_2) - x(t_1)$$

So A is the **displacement** of the particle during the time from t_1 to t_2.

When the graph is above and below the time axis

Let A_1 be the area above the time axis and A_2 the area below it (Figure 7.6).

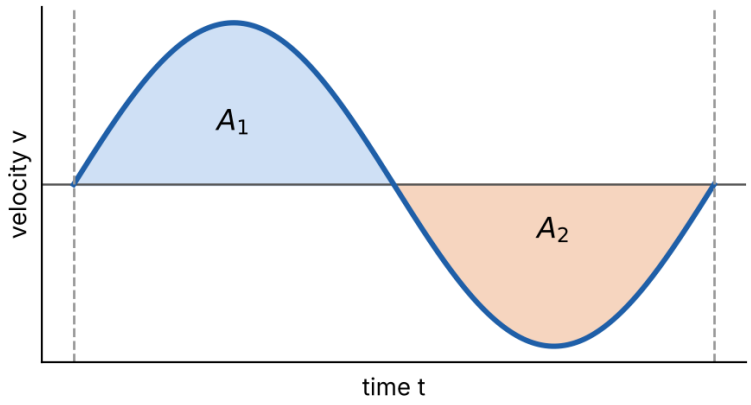
Note

Displacement = $A_1 - A_2$

Distance = $A_1 + A_2$

Why? Area below the axis means the particle moves backward, so it is *subtracted* for displacement. Distance counts every part of the path as positive, so both areas are *added*.

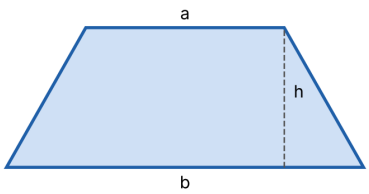
Figure 7.6 Displacement = $A_1 - A_2$, Distance = $A_1 + A_2$



Area of a trapezium

A trapezium (trapezoid) is a four-sided plane figure with **exactly one pair of parallel sides**. If a and b are the lengths of the parallel sides and h is the perpendicular distance between them, then:

$$\text{Area} = \tfrac{1}{2}(a+b) \times h$$



Many velocity-time graphs enclose a trapezium, so this formula is used again and again.

3. Motion with constant acceleration

To study this motion we use four quantities: displacement, time, velocity and acceleration. Usually some are known and the rest must be found. The **equations of motion** connect them.

Setup. A particle moves along a straight line with constant acceleration a. At time t = 0 it is at the point 0 (that is, x = 0) and has initial velocity u.

First equation of motion

Acceleration is the rate of change of velocity with time. Multiply both sides by dt.

$$\frac{dv}{dt} = a \;\Rightarrow\; dv = a\,dt$$

Integrate both sides. Here c is the constant of integration.

$$\int dv = \int a\,dt \;\Rightarrow\; v = at + c \qquad \ldots(1)$$

Use the starting condition: at t = 0 the velocity is v = u. Put these in equation (1).

$$u = 0 + c \;\Rightarrow\; c = u$$

Put c back into equation (1). This is the **1st equation of motion**.

$$v = u + at \qquad \ldots(2)$$

Second equation of motion

Velocity is the rate of change of displacement. Put in v = u + at from equation (2).

$$v = \frac{dx}{dt} \;\Rightarrow\; \frac{dx}{dt} = u + at \;\Rightarrow\; dx = (u + at)\,dt$$

Integrate both sides. The integral of at is at²/2, and c_1 is a new constant.

$$\int dx = \int (u + at)\,dt \;\Rightarrow\; x = ut + a\frac{t^2}{2} + c_1 \qquad \ldots(3)$$

Use the starting condition: at t = 0 the position is x = 0. Put these in equation (3).

$$0 = 0 + 0 + c_1 \;\Rightarrow\; c_1 = 0$$

Put c_1 back into equation (3). This is the **2nd equation of motion**.

$$x = ut + \tfrac{1}{2}at^2 \qquad \ldots(4)$$

Third equation of motion

From equation (2), find t. This lets us remove time from the equations.

$$t = \frac{v - u}{a}$$

Substitute this t into equation (4) to eliminate t.

$$x = u\left(\frac{v - u}{a}\right) + \tfrac{1}{2}a\left(\frac{v - u}{a}\right)^2$$

Simplify the second term. One factor of a cancels.

$$x = \frac{u(v - u)}{a} + \frac{(v - u)^2}{2a}$$

Multiply both sides by 2a to remove the denominators.

$$2ax = 2u(v - u) + (v - u)^2$$

Take (v − u) common from both terms.

$$2ax = (v - u)[\,2u + (v - u)\,] = (v - u)(v + u)$$

Use (v − u)(v + u) = v² − u². This is the **3rd equation of motion**.

$$2ax = v^2 - u^2$$

The three equations of motion

For a particle moving with constant acceleration:

$$v = u + at \qquad x = ut + \tfrac{1}{2}at^2 \qquad 2ax = v^2 - u^2$$

These equations are used often to describe and analyse motion under constant acceleration.

Exercise 7: Solved Questions

Q1 Kamran's journey

EXERCISE 7

QUESTION

Kamran covers 100 km in the first 2 hours, stops for 1 hour, and then drives 60 km. The displacement-time graph is shown in Figure 7.9. (a) Find his average velocity. (b) Find his velocity during the last phase of the journey.

NOTE The question text says “the next 2 hours”, but the solution in the notes, and the graph (which ends at 6 hours), use 3 hours for the last phase. We follow the solution.

HINT

Average velocity = total displacement ÷ total time. The stop counts in the total time, but its displacement is 0. For the last phase, divide only that phase's displacement by that phase's time.

اردو ہنٹ

اوسط velocity نکالنے کے لیے کل displacement کو کل وقت پر تقسیم کریں۔ جب کامران رکا ہوا ہو تو displacement صفر ہوتا ہے، مگر وہ ایک گھنٹہ کل وقت میں ضرور شامل ہوتا ہے۔ آخری مرحلے کی velocity = اُس مرحلے کا displacement ÷ اُس مرحلے کا وقت۔

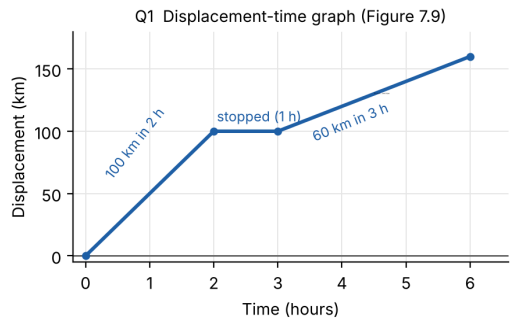


Figure 7.9: displacement-time graph of Kamran

FORMULAS USED

- Average velocity = total displacement ÷ total time
- Velocity = displacement ÷ time ($v = \Delta s \div \Delta t$)

SOLUTION

- **(a) Average velocity**

  Using average velocity = total displacement ÷ total time:

  Total time = 2 + 1 + 3 = 6 h

  Total displacement = 100 + 0 + 60 = 160 km

  $v_{avg} = \frac{160}{6} = 26.67$ km/h

- **(b) Velocity in the last phase**

  Using velocity = displacement ÷ time: $v = \frac{60}{3} = 20$ km/h

ANSWER **(a) 26.67 km/h. (b) 20 km/h.**

Q2 A car that changes direction

EXERCISE 7

QUESTION

The displacement-time graph of a moving car is shown in Figure 7.10. The car changes its direction during the journey and its motion has three distinct phases. (a) What is the car's velocity in each phase? (b) When did the car pass its starting point again? (c) What is the total distance travelled? (d) What is the average velocity?

HINT

On a displacement-time graph, velocity = slope = $(y_2 - y_1) \div (x_2 - x_1)$. A negative slope means the car is moving backward. For distance, add the length of every phase as a positive number. For average velocity, use only the start and end positions.

اردو ہنٹ

displacement-time graph میں velocity = slope ہوتی ہے۔ ہر مرحلے کے دو points لے کر $(y_2 - y_1) \div (x_2 - x_1)$ slope = نکالیں۔ منفی slope کا مطلب ہے گاڑی پیچھے کی طرف جا رہی ہے۔ کل فاصلہ نکالتے وقت ہر مرحلے کا فاصلہ مثبت لے کر جمع کریں، جبکہ displacement میں صرف شروع اور آخر کی پوزیشن دیکھی جاتی ہے۔

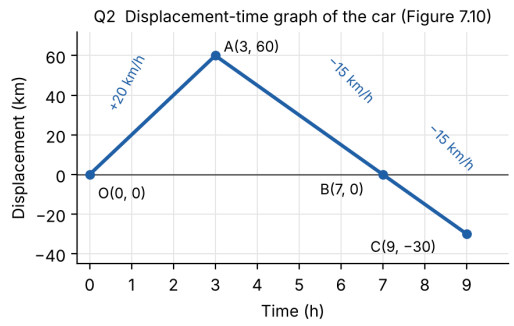


Figure 7.10: displacement-time graph of the car

FORMULAS USED

- Velocity = slope of displacement-time graph = $(y_2 - y_1) \div (x_2 - x_1)$
- Distance = sum of all path lengths (each taken positive)
- Average velocity = total displacement ÷ total time

SOLUTION

- **(a) Velocity in each phase**

  Using velocity = slope. Phase 1, O(0, 0) to A(3, 60): $v = \frac{60 - 0}{3 - 0} = \frac{60}{3} = 20$ km/h

  Phase 2, A(3, 60) to B(7, 0): $v = \frac{0 - 60}{7 - 3} = \frac{-60}{4} = -15$ km/h

  Phase 3, B(7, 0) to C(9, −30): $v = \frac{-30 - 0}{9 - 7} = \frac{-30}{2} = -15$ km/h

- **(b) Passing the starting point again**

  Displacement = 0 at B(7, 0), so t = 7 h

- **(c) Total distance**

  Using distance = sum of all phases (positive): 60 + 60 + 30 = 150 km

- **(d) Average velocity**

  Total displacement = final − start = −30 − 0 = −30 km (that is, 60 − 90 = −30 km)

  Total time = 9 h

  $v_{avg} = \frac{-30}{9} = -3.33$ km/h

ANSWER **(a) 20 km/h, −15 km/h, −15 km/h. (b) After 7 hours. (c) 150 km. (d) −3.33 km/h.**

Q3 Aslam runs along a track

EXERCISE 7

QUESTION

Figure 7.11 is the velocity-time graph of Aslam's motion as he runs along a straight track. For the first 5 seconds he accelerates uniformly from rest to a velocity of 8 m/s. He then keeps this velocity for a further 8 seconds. Find (a) the rate at which Aslam accelerates, and (b) his displacement from the starting point after 13 seconds.

HINT

On a velocity-time graph, acceleration = slope = rise ÷ run. Displacement = area under the graph. Here the shape under the graph is a trapezium with area ½(a + b) × h, where a and b are the parallel sides and h is the height between them.

اردو ہنٹ

velocity-time graph میں acceleration = slope ہوتی ہے، یعنی اونچائی (Rise) کو لمبائی (Run) پر تقسیم کریں۔ Displacement graph کے نیچے والے رقبے کے برابر ہوتی ہے، اور یہاں شکل trapezium ہے: ½ (a + b) × h۔ یاد رہے a اور b آپس میں متوازی اطراف ہیں اور h ان کے درمیان سیدھی اونچائی ہے۔

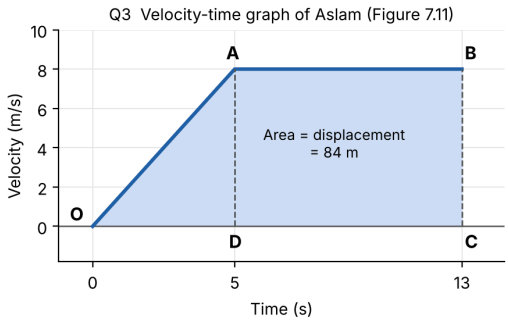


Figure 7.11: velocity-time graph of Aslam

FORMULAS USED

- Acceleration = slope of velocity-time graph = rise ÷ run
- Displacement = area under the velocity-time graph
- Area of trapezium = ½ (a + b) × h

SOLUTION

- **(a) Acceleration**

  Using acceleration = slope: a = slope of OA = $\frac{\text{Rise}}{\text{Run}} = \frac{8}{5} = 1.6$ m/s²

- **(b) Displacement after 13 s**

  Parallel sides: a = |AB| = 13 − 5 = 8, b = |OC| = 13. Height: h = |AD| = 8

  Using trapezium formula: displacement = area of trapezium OABC = $\frac{1}{2}(a + b)h$

  $= \frac{1}{2}(8 + 13) \times 8 = 21 \times 4 = 84$ m

ANSWER **(a) a = 1.6 m/s². (b) Displacement = 84 m.**

Q4 A car that brakes

EXERCISE 7

QUESTION

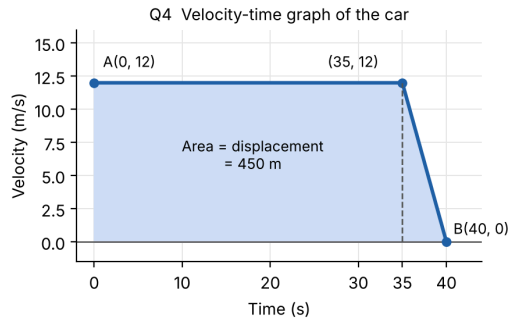
A car is travelling along a straight road. At t = 0 s the car passes a point A with a velocity of 12 m/s, and this velocity is maintained up to t = 35 s. The driver then applies the brakes and the car decelerates uniformly, coming to rest at the point B at t = 40 s. (a) Sketch a velocity-time graph for the motion. (b) Find the displacement from A to B.

HINT

Plot the points (0, 12), (35, 12) and (40, 0), with time on the x-axis and velocity on the y-axis. Constant velocity gives a horizontal line. Uniform deceleration gives a straight line going down to 0. The displacement is the area of the trapezium under the graph.

اردو ہنٹ

پہلے تین points لگائیں: (0, 12)، (35, 12) اور (40, 0)۔ 35 سیکنڈ تک سیدھی افقی لکیر کھینچیں (کیونکہ velocity مستقل ہے) اور پھر نیچے آتی ہوئی سیدھی لکیر B تک (کیونکہ رفتار یکساں طور پر کم ہو رہی ہے)۔ Displacement = graph کے نیچے والے trapezium کا رقبہ۔



Velocity-time graph of the car; the shaded area is the displacement

FORMULAS USED

- Displacement = area under the velocity-time graph
- Area of trapezium = ½ (a + b) × h

SOLUTION

- **(a) Velocity-time graph**

  Points: A(0, 12), (35, 12), B(40, 0)

  A to (35, 12): horizontal line. (35, 12) to B(40, 0): straight line going down.

  The sketch is shown above.

- **(b) Displacement from A to B**

  Parallel sides: a = |AP| = 35, b = |OB| = 40. Height: h = |PQ| = 12

  Using trapezium formula: displacement = area of trapezium OAPB = $\frac{1}{2}(a + b)h$

  $= \frac{1}{2}(35 + 40) \times 12 = 75 \times 6 = 450$ m

ANSWER **Displacement from A to B = 450 m.**

Q5 An object moving for 12 seconds

EXERCISE 7

QUESTION

Figure 7.12 shows the velocity-time graph of an object moving in a straight line over a 12-second time interval. (a) At what time did the object change its direction of motion? (b) Calculate the total distance travelled and the displacement of the object during the 12 seconds.

HINT

The direction changes where the graph crosses the time axis (velocity changes sign). Find the area above the axis (A_1) and the area below it (A_2) separately. Displacement = $A_1 - A_2$. Distance = $A_1 + A_2$.

اردو ہنٹ

جہاں graph وقت کے محور کو کاٹتا ہے وہیں سمت بدلتی ہے۔ محور کے اوپر والا رقبہ A_1 اور نیچے والا رقبہ A_2 الگ الگ نکالیں۔ Displacement = $A_1 - A_2$ (نیچے والا رقبہ منفی) اور فاصلہ Distance = $A_1 + A_2$ (دونوں رقبے مثبت جمع ہوتے ہیں)۔

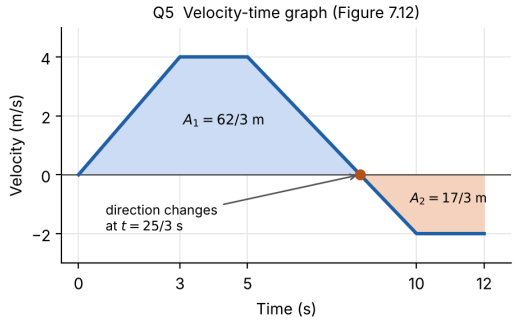


Figure 7.12: velocity-time graph of the object

FORMULAS USED

- Direction changes where the graph crosses the time axis (v = 0)
- Area of trapezium = ½ (a + b) × h
- Displacement = $A_1 - A_2$
- Distance = $A_1 + A_2$

SOLUTION

- **(a) Time of change of direction**

  Graph crosses the time axis at $t = \frac{25}{3}$ s ≈ 8.33 s

- **(b) Distance and displacement**

  A_1 = trapezium OABC: a = 5 − 3 = 2, b = $\frac{25}{3}$, $h_1 = 4$

  $A_1 = \frac{1}{2}(2 + \frac{25}{3}) \times 4 = \frac{31}{3} \times 2 = \frac{62}{3}$ m

  A_2 = trapezium CDEF: c = 12 − 10 = 2, d = 12 − $\frac{25}{3}$ = $\frac{11}{3}$, $h_2 = 2$

  $A_2 = \frac{1}{2}(2 + \frac{11}{3}) \times 2 = 2 + \frac{11}{3} = \frac{17}{3}$ m

  Displacement = $A_1 - A_2 = \frac{62}{3} - \frac{17}{3} = \frac{45}{3} = 15$ m

  Distance = $A_1 + A_2 = \frac{62}{3} + \frac{17}{3} = \frac{79}{3} = 26.33$ m

ANSWER **(a) t = 25/3 s ≈ 8.33 s. (b) Displacement = 15 m and distance = 26.33 m.**

Q6 A vehicle stopped by brakes

EXERCISE 7

QUESTION

A vehicle moving with a speed of 36 km/h is stopped by applying brakes and comes to rest in 10 seconds. Find the retardation and the distance covered during this time.

HINT

First change km/h into m/s (× 1000 ÷ 3600). The final velocity is v = 0 because the vehicle stops. Find a from v = u + at, then find the distance from $2ax = v^2 - u^2$. Retardation means the negative of the acceleration.

اردو ہنٹ

سب سے پہلے رفتار کو km/h سے m/s میں بدلیں (36 × 1000 ÷ 3600)۔ رکنے پر آخری رفتار v = 0 ہوتی ہے۔ پہلی مساوات v = u + at سے a نکالیں، پھر تیسری مساوات $2ax = v^2 - u^2$ سے فاصلہ نکالیں۔ Retardation کا مطلب acceleration کی منفی قدر ہے۔

FORMULAS USED

- Convert km/h to m/s: × 1000 ÷ 3600
- 1st equation of motion: v = u + at
- 3rd equation of motion: $2ax = v^2 - u^2$
- Retardation = − acceleration

SOLUTION

$u = 36 \text{ km/h} = \frac{36 \times 1000}{3600} \text{ m/s} = 10 \text{ m/s}$, $v = 0$, $t = 10 \text{ s}$

1st equation: $v = u + at \rightarrow 0 = 10 + a(10) \rightarrow 10a = -10 \rightarrow a = -1 \text{ m/s}^2$

Retardation $= -a = -(-1) = 1 \text{ m/s}^2$

3rd equation: $2ax = v^2 - u^2 \rightarrow 2(-1)x = 0^2 - 10^2 \rightarrow -2x = -100$

$x = \frac{-100}{-2} = 50 \text{ m}$

ANSWER **Retardation = 1 m/s² and distance = 50 m.**

Q7 A train speeding up

EXERCISE 7

QUESTION

A train moving at 54 km/h is accelerated uniformly to 90 km/h in 20 s. Find the acceleration and the distance covered in this time.

HINT

Change both speeds from km/h into m/s (divide by 3.6). Find a from $v = u + at$. Then put the values into $2ax = v^2 - u^2$ to find the distance.

اردو ہنٹ

دونوں رفتاروں کو km/h سے m/s میں بدلیں (یعنی 3.6 سے تقسیم کریں)۔ پہلے $v = u + at$ سے acceleration نکالیں، پھر $2ax = v^2 - u^2$ میں قدریں رکھ کر فاصلہ معلوم کریں۔

FORMULAS USED

- Convert km/h to m/s: × 1000 ÷ 3600
- 1st equation of motion: $v = u + at$
- 3rd equation of motion: $2ax = v^2 - u^2$

SOLUTION

$u = 54 \times \frac{1000}{3600} = 15$ m/s, $v = 90 \times \frac{1000}{3600} = 25$ m/s, $t = 20$ s

1st equation: $v = u + at \rightarrow 25 = 15 + a(20) \rightarrow 20a = 10$

$a = \frac{10}{20} = 0.5$ m/s²

3rd equation: $2ax = v^2 - u^2 \rightarrow 2(0.5)x = 25^2 - 15^2 = 625 - 225$

$x = 400$ m

ANSWER **Acceleration = 0.5 m/s² and distance = 400 m.**

Q8 Distance in the last three seconds

EXERCISE 7

QUESTION

A particle moving in a straight line starts from rest and is accelerated uniformly to attain a velocity of 30 km/h in 5 seconds. Find the distance travelled by the particle in the last three seconds.

HINT

Starting from rest means u = 0. Change 30 km/h into m/s and find a from v = u + at. Distance in the last 3 s = (distance in all 5 s) − (distance in the first 2 s), using x = ut + ½at² each time.

اردو ہنٹ

آخری 3 سیکنڈ کا فاصلہ = (پورے 5 سیکنڈ کا فاصلہ) − (پہلے 2 سیکنڈ کا فاصلہ)۔ ذرّہ ساکن سے شروع ہوتا ہے اس لیے $u = 0$۔ پہلے $v = u + at$ سے a نکالیں، پھر دونوں فاصلے $x = ut + \frac{1}{2}at^2$ سے الگ الگ نکال کر تفریق کریں۔

FORMULAS USED

- Convert km/h to m/s: × 1000 ÷ 3600
- 1st equation of motion: $v = u + at$
- 2nd equation of motion: $x = ut + \frac{1}{2}at^2$
- Distance in last 3 s = $x_5 - x_2$

SOLUTION

$u = 0$, $t = 5$ s, $v = 30$ km/h $= \frac{30 \times 1000}{3600} = \frac{25}{3}$ m/s

1st equation: $v = u + at \rightarrow \frac{25}{3} = 0 + a(5) \rightarrow a = \frac{25}{3 \times 5} = \frac{5}{3}$ m/s²

2nd equation: $x_5 = ut + \frac{1}{2}at^2 = (0)(5) + \frac{1}{2}(\frac{5}{3})(5)^2 = \frac{125}{6} = 20.83$ m

$x_2 = (0)(2) + \frac{1}{2}(\frac{5}{3})(2)^2 = \frac{20}{6} = 3.33$ m

$x = x_5 - x_2 = 20.83 - 3.33 = 17.5$ m

ANSWER **Distance travelled in the last three seconds = 17.5 m.**

Q9 Two particles meet

QUESTION

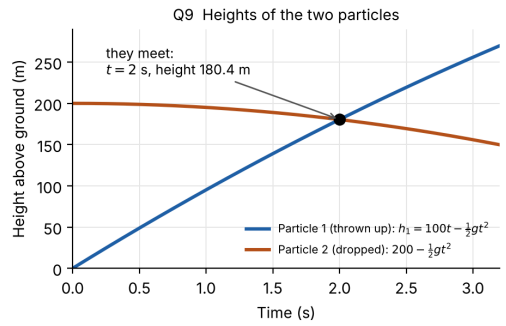
A particle is projected vertically upward with a velocity of 100 m/s, and another is let fall from a height of 200 m at the same time. Find the height of the point where they meet each other.

HINT

Take g = 9.8 m/s². Write the distance moved by each particle at time t using x = ut + ½at². When they meet, the distance gone up by particle 1 plus the distance fallen by particle 2 equals 200 m. The ½gt² terms cancel, so you can find t, then the height.

اردو ہنٹ

دونوں ذرّوں کی اونچائی الگ الگ لکھیں۔ اوپر پھینکے گئے ذرّے کے لیے $u_1 = 100$ اور $a_1 = -g$ ہے، جبکہ گرائے گئے ذرّے کے لیے $u_2 = 0$ اور $a_2 = g$۔ جب وہ ملیں گے تو دونوں کے طے کیے ہوئے فاصلوں کا مجموعہ $h_1 + h_2 = 200$ ہوگا۔ $\frac{1}{2}gt^2$ والے دونوں رقم کٹ جاتے ہیں، اس سے t مل جاتا ہے۔ پھر t کی قدر h_1 میں رکھ کر اونچائی نکالیں۔



Heights of the two particles above the ground

FORMULAS USED

- 2nd equation of motion: x = ut + ½at² (a = −g going up, a = +g coming down)
- When they meet: h₁ + h₂ = 200

SOLUTION

Particle 1 (up): $u_1 = 100$, $a_1 = -g$

2nd equation: $h_1 = u_1 t + \frac{1}{2}a_1 t^2 = 100t - \frac{1}{2}gt^2$

Particle 2 (dropped): $u_2 = 0$, $a_2 = g$

$h_2 = 0 \cdot t + \frac{1}{2}gt^2 = \frac{1}{2}gt^2$

$h_1 + h_2 = 200 \rightarrow 100t - \frac{1}{2}gt^2 + \frac{1}{2}gt^2 = 200$

$100t = 200 \rightarrow t = 2$ s

$h_1 = 100(2) - \frac{1}{2}(9.8)(2)^2 = 200 - 19.6 = 180.4$ m

ANSWER **The particles meet after 2 s, at a height of 180.4 m.**

Q10 A ball thrown upward

EXERCISE 7

QUESTION

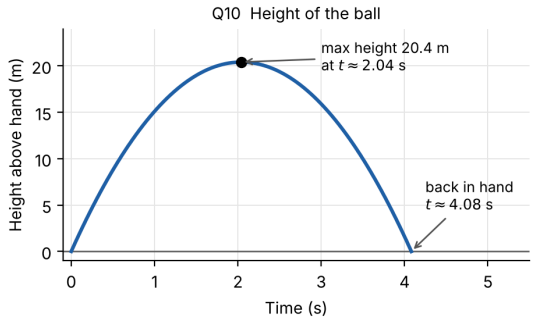
A ball is thrown vertically upward with a velocity of 20 m/s. Find the maximum height reached and the time taken to return to the thrower's hand.

HINT

At the highest point the ball stops for a moment, so v = 0, and the acceleration is a = −g = −9.8 m/s² while going up. Use 2ax = v² − u² for the maximum height and v = u + at for the time to the top. The time down equals the time up, so the total time is double.

اردو ہنٹ

سب سے اونچے مقام پر گیند لمحہ بھر کے لیے رکتی ہے، یعنی $v = 0$، اور اوپر جاتے ہوئے $a = -g$ ہوتا ہے۔ زیادہ سے زیادہ اونچائی $2ax = v^2 - u^2$ سے ملتی ہے۔ اوپر جانے کا وقت $v = u + at$ سے نکالیں۔ نیچے آنے کا وقت بھی اتنا ہی ہوتا ہے، اس لیے واپسی کا کل وقت اس سے دُگنا ہوگا۔



Height of the ball against time

FORMULAS USED

- 3rd equation of motion: 2ax = v² − u² (a = −g going up)
- 1st equation of motion: v = u + at
- Return time = 2 × time to reach the top

SOLUTION

- **Maximum height**

  u = 20 m/s, v = 0, a = −g = −9.8 m/s²

  3rd equation: 2ax = v² − u² → −2(9.8)x = 0² − 20² → −19.6x = −400

  $x = \frac{400}{19.6} = 20.4$ m

- **Time to return**

  1st equation: v = u + at → 0 = 20 − gt → 9.8t = 20 → $t = \frac{20}{9.8} \approx 2.04$ s

  Return time = 2 × 2.04 ≈ 4.08 s

  (The notes write 4.085 s; the exact value 40 ÷ 9.8 = 4.08 s to two decimal places.)

ANSWER **Maximum height = 20.4 m and time to return ≈ 4.08 s.**

Q11 Two rocks dropped from a cliff

EXERCISE 7

QUESTION

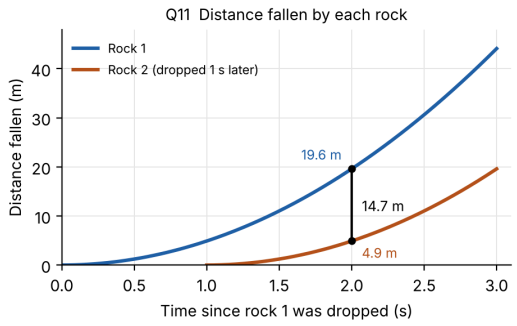
A rock is dropped from a cliff. One second later, another rock is dropped from the same height. (a) What is the distance between the two rocks 2 seconds after the first is dropped? (b) At what time will the distance between them be 14.7 m?

HINT

A rock dropped from rest has u = 0, so x = ½gt². The second rock falls for one second less than the first, so use (t − 1) for it. In part (b), subtract the two distances and set the result equal to 14.7. The t² terms cancel and a simple equation is left.

اردو ہنٹ

چٹان کو چھوڑا جائے تو $u = 0$، لہٰذا فاصلہ $x = \frac{1}{2}gt^2$ ہوگا۔ دوسری چٹان ایک سیکنڈ بعد گرائی گئی، اس لیے جب پہلی t سیکنڈ گری ہو تو دوسری صرف $(t-1)$ سیکنڈ گری ہوتی ہے۔ حصہ (b) میں فرق $\frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2 = 14.7$ لکھیں؛ t^2 کٹ جاتا ہے اور سادہ مساوات رہ جاتی ہے۔



Distance fallen by each rock

FORMULAS USED

- 2nd equation of motion: $x = ut + \frac{1}{2}at^2$, with $u = 0 \rightarrow x = \frac{1}{2}gt^2$
- Distance between rocks = $x_1 - x_2$

SOLUTION

2nd equation, with u = 0: $x = ut + \frac{1}{2}gt^2 = \frac{1}{2}gt^2$ …(1) ($g = 9.8\ m/s^2$)

- **(a) Distance between the rocks at t = 2 s**

$x_1 = \frac{1}{2}(9.8)(2)^2 = 19.6$ m (rock 1 falls for 2 s)

$x_2 = \frac{1}{2}(9.8)(1)^2 = 4.9$ m (rock 2 falls for 1 s)

$x_1 - x_2 = 19.6 - 4.9 = 14.7$ m

- **(b) When is the distance 14.7 m?**

$\Delta x = \frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2 = 14.7$

$14.7 = \frac{1}{2}g(t^2 - t^2 + 2t - 1) = \frac{1}{2}g(2t-1)$

$14.7 \times 2 = g(2t-1) \rightarrow 29.4 = 9.8(2t-1)$

$2t - 1 = \frac{29.4}{9.8} = 3 \rightarrow 2t = 4 \rightarrow t = 2$ s

ANSWER **(a) 14.7 m. (b) t = 2 s after the first rock is dropped.**

Q12 Velocity from a changing acceleration

EXERCISE 7

QUESTION

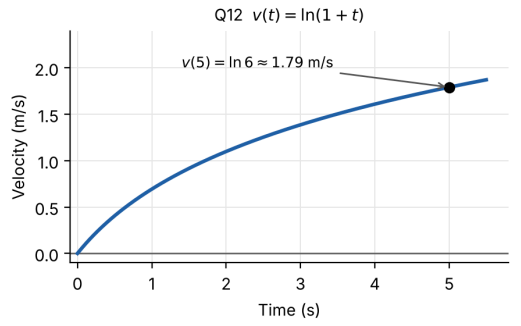
The acceleration of a particle is a(t) = 1/(1 + t) for t ≥ 0. If the velocity is 0 m/s when t = 0, find the velocity at t = 5 seconds.

HINT

Acceleration = dv/dt, so write dv = a dt and integrate both sides. Remember that the integral of 1/(1 + t) dt is ln(1 + t). Find the constant c from the starting condition (v = 0 at t = 0), and finally put t = 5.

اردو ہنٹ

acceleration = dv/dt ہوتی ہے، اس لیے dv = a dt لکھ کر دونوں طرف integrate کریں۔ ∫ 1/(1 + t) یاد رکھیں۔ dt = ln(1 + t) پھر ابتدائی شرط (v = 0 پر t = 0) سے مستقل c نکالیں، اور آخر میں t = 5 رکھ کر جواب لیں۔



Graph of v(t) = ln(1 + t)

FORMULAS USED

- Acceleration = dv/dt → dv = a dt
- ∫ 1/(1 + t) dt = ln(1 + t) + c
- Starting condition: v = 0 at t = 0

SOLUTION

$$a(t) = \frac{1}{1+t} \rightarrow \frac{dv}{dt} = \frac{1}{1+t} \rightarrow dv = \frac{1}{1+t}\,dt$$

$$\int dv = \int \frac{1}{1+t}\,dt \rightarrow v = \ln(1+t) + c \quad \ldots(1)$$

$$t = 0,\ v = 0:\ 0 = \ln(1+0) + c = \ln 1 + c = 0 + c \rightarrow c = 0$$

$$v(t) = \ln(1+t)$$

$$v(5) = \ln(1+5) = \ln 6 \approx 1.79 \text{ m/s}$$

ANSWER **The velocity at t = 5 s is about 1.79 m/s.**

Q13 Distance travelled by a robotic arm

EXERCISE 7

QUESTION

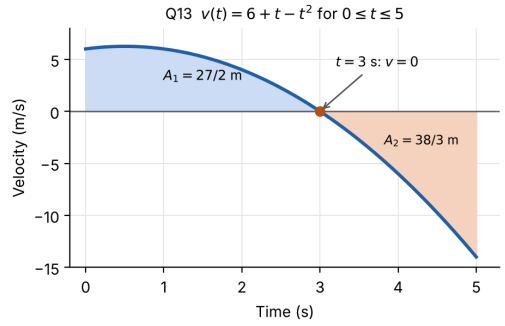
The motion of a robotic arm moving along a straight track is described by the velocity function $v(t) = 6 + t - t^2$, where $v(t)$ is in metres per second and t is in seconds, valid for $0 \le t \le 5$. Find the distance travelled by the arm.

HINT

Distance is always positive, so first find the time when $v = 0$ and split the interval there. Integrate on each part. Take the absolute value of the part where v is negative, then add the two parts.

اردو ہنٹ

فاصلہ ہمیشہ مثبت ہوتا ہے، اس لیے پہلے وہ وقت نکالیں جہاں $v = 0$ ہو۔ $v = 0 \Rightarrow (3 - t)(2 + t) = 0$ سے $t = 3$ ملتا ہے ($t = -2$ وقفے $[0, 5]$ سے باہر ہے)۔ وقت کو $[0, 3]$ اور $[3, 5]$ میں بانٹیں۔ دوسرے حصے کے integral کی absolute value لیں اور پھر دونوں حصے جمع کریں۔



$v(t) = 6 + t - t^2$: the area above the axis is A_1 and the area below is A_2

FORMULAS USED

- Distance $= \int |v|\, dt = A_1 + A_2$ (split where $v = 0$)
- $\int t^n\, dt = t^{n+1} \div (n + 1)$
- Find t where $v = 0$ by factorising

SOLUTION

- **Find where $v = 0$**

  $6 + t - t^2 = 0 \rightarrow 6 + 3t - 2t - t^2 = 0$

  $3(2 + t) - t(2 + t) = 0 \rightarrow (3 - t)(2 + t) = 0$

  $t = 3$ ($t = -2$ is not in $[0, 5]$, so reject it). A_1 is on $[0, 3]$ and A_2 is on $[3, 5]$.

- **Integrate**

  $F(t) = 6t + \frac{t^2}{2} - \frac{t^3}{3}$, Distance $= F(3) - F(0) + |F(5) - F(3)|$

  $F(3) = 6(3) + \frac{3^2}{2} - \frac{3^3}{3} = 18 + \frac{9}{2} - 9 = \frac{27}{2}$

  $A_1 = \frac{27}{2} - 0 = \frac{27}{2}$ m

  $F(5) = 6(5) + \frac{25}{2} - \frac{125}{3} = \frac{180 + 75 - 250}{6} = \frac{5}{6}$

  $\int_3^5 v\, dt = \frac{5}{6} - \frac{27}{2} = \frac{5 - 81}{6} = -\frac{38}{3}$

  $A_2 = \left|-\frac{38}{3}\right| = \frac{38}{3}$ m

  Distance $= \frac{27}{2} + \frac{38}{3} = \frac{81 + 76}{6} = \frac{157}{6} = 26.17$ m

ANSWER **The arm travels 26.17 m.**

Quick Revision Sheet

What you need	Formula or rule
Average velocity	Total displacement ÷ total time
Average speed	Total distance ÷ total time
Velocity from a displacement-time graph	Slope of the graph (positive = forward, negative = backward)
Acceleration from a velocity-time graph	Slope of the graph (dv/dt)
Displacement from a velocity-time graph	Area under the graph. Use $A_1 - A_2$ if part of it is below the axis
Distance from a velocity-time graph	$A_1 + A_2$ (add all areas as positive)
Area of a trapezium	$\frac{1}{2}(a + b) \times h$
1st equation of motion	$v = u + at$
2nd equation of motion	$x = ut + \frac{1}{2}at^2$
3rd equation of motion	$2ax = v^2 - u^2$
Convert km/h to m/s	Multiply by 1000 and divide by 3600 (that is, divide by 3.6)
Up and down motion (free fall)	Use the same equations with $a = -g$ going up and $a = +g$ coming down. $g = 9.8\ m/s^2$
Velocity or position from calculus	$v = \int a\,dt$ and $x = \int v\,dt$. Then use the starting condition to find the constant

Things to remember

- Write the given values with units first, and change km/h into m/s before using the equations.
- Choose one direction as positive. Upward motion with gravity: $a = -g$.
- Displacement can be zero or negative. Distance is never negative.
- If a velocity-time graph goes below the axis, find the two areas separately.
- Always check the answer: the units and the sign should make sense.