Unit 11 Vector Valued Functions and Their Derivatives

Mathematics-12 · Second Year · Chapter Notes · Prepared by Muhammad Ali Photostate Zafarwal

ہدایت: ہر سوال کے شروع میں سبز خانے میں اردو ہدایت دی گئی ہے کہ سوال میں what to do — یعنی کون سا طریقہ اور کون سی ترتیب استعمال کرنی ہے۔

CONCEPT
Yellow box on the side appears next to the exact step where a concept / formula is used.

11.1 Scalar and Vector-Valued Functions of a Single Variable

11.1.1 Scalar Function

CONCEPT
Scalar function: **output is a single real number** (one value only).

A scalar function is a function that assigns a **single real number** (a scalar) to each point in its domain. Think of it as a rule that assigns one value to a position in space, like measuring temperature at a specific point.

Mathematically, if to each value of a scalar variable t in the domain of a function $f(t)$, there corresponds a unique scalar f, then f is called a **scalar function** of t. A real-valued function is commonly called a scalar function.

Example: $f(t) = 3t + 2\cos t$ is a scalar function.

11.1.2 Vector-Valued Function (Vector Function)

CONCEPT
Vector function: a function whose **output is a vector** (two or three components).

A vector-valued function in a single variable is a function that takes **one variable as input** and returns a **vector as output**. Each component of the output vector is itself a scalar function of the input. Typically, a vector-valued function is written as:

$$\vec{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the x-, y- and z-axes, and the scalar functions $f(t), g(t)$ and $h(t)$ are the **components** of $\vec{r}(t)$.

11.1.3 Constructing a Vector-Valued Function

CONCEPT
Parametric equations $x = f(t),\ y = g(t),\ z = h(t)$ become a vector function when each is attached to $\hat{i}, \hat{j}, \hat{k}$.

Step 1: Determine the Components. Identify how each coordinate changes with respect to t.

Step 2: Combine into Vector Form. Express the components as a single vector using either component notation $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ or unit vectors $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$.

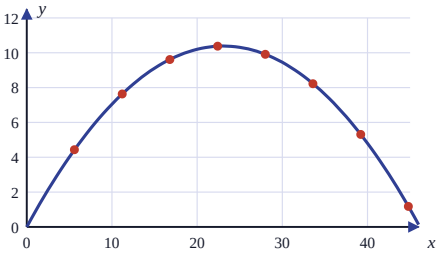
Step 3: Specify the Domain. Define the range of t where the function is valid (e.g., $t \geq 0$) for time-dependent motion.

Example 1: Projectile Motion

When a ball is thrown at an angle, its position over time follows a parabolic path under the influence of gravity. If the initial velocity is v_0 at an angle θ, the position vector is:

$$\vec{r}(t) = (v_0 \cos\theta)\, t\, \hat{i} + \left((v_0 \sin\theta)\, t - \tfrac{1}{2} g t^2\right)\hat{j}$$

where $(v_0 \cos\theta)t$ is the x-component (horizontal motion at constant speed) and $(v_0 \sin\theta)t - \frac{1}{2}gt^2$ is the y-component (vertical motion with gravity deceleration). This model simulates trajectories of objects like soccer balls, arrows or rockets in 2D.



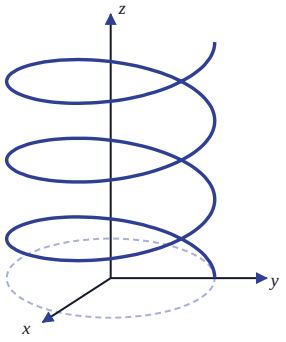
Projectile path (positions at 0.5 s steps)

Example 2: Flight Path of a Drone

A drone taking off and moving in a helical search pattern can be modeled as:

$$\vec{r}(t) = 5\cos t\, \hat{i} + 5\sin t\, \hat{j} + 0.5t\, \hat{k}$$

The x- and y-components represent the circular motion with a 5-metre radius, and the z-component represents the steady ascent at 0.5 metres per second. This describes real-world applications like surveillance, search missions, or aerial photography.



Helical path of the drone

11.2 Domain and Range of a Vector-Valued Function

The **domain** of a vector-valued function $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ is the set of all real numbers t for which **every** component function $f(t), g(t)$ and $h(t)$ is defined.

The **range** of a vector-valued function is the set of all possible output vectors $\vec{r}(t)$ as t varies over the domain.

CONCEPT

Domain of $\vec{r}$ = intersection of the domains of its components. Range = all output vectors.

11.2.1 Steps to Find the Domain of a Vector-Valued Function

Step 1: Determine the domain of each component separately.

Step 2: Find the intersection of the domains of each component $f(t), g(t)$ and $h(t)$. The domain of $\vec{r}(t)$ is the intersection of the domains of $f(t), g(t)$ and $h(t)$.

DOMAIN CHECKLIST

$\sqrt{u}$: need $u \geq 0$

$\ln u$: need $u > 0$

$\dfrac{1}{u}$: need $u \neq 0$

$\sin t, \cos t, e^t, t^n$: all real t

Example 1 **Find the domain and range of the vector-valued function $\vec{r}(t) = \sqrt{4-t^2}\,\hat{i} + \dfrac{1}{t-1}\,\hat{j}$.**

ہدایت: پہلے دونوں components کا الگ الگ domain نکالیں: جڑ کے اندر کی مقدار ≥ 0 ہونی چاہیے اور کسر کا denominator $\neq 0$ ہونا چاہیے۔ پھر دونوں domains کا intersection لیں۔ آخر میں range کو $\langle x(t), y(t)\rangle$ کی شکل میں لکھیں۔

Solution:

Domain of each component.

$x(t) = \sqrt{4-t^2}$ is defined when $4 - t^2 \geq 0$.

$$4 - t^2 \geq 0 \;\Rightarrow\; t^2 \leq 4 \;\Rightarrow\; -2 \leq t \leq 2$$

$$D_x = [-2,\, 2]$$

$y(t) = \dfrac{1}{t-1}$ is defined when $t - 1 \neq 0$, i.e. $t \neq 1$.

$$D_y = \mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, \infty)$$

Intersection.

$$\text{Domain} = D_x \cap D_y = [-2, 2] \cap \big((-\infty, 1) \cup (1, \infty)\big) = [-2, 1) \cup (1, 2]$$

Range. All vectors obtained for t in the domain:

$$\text{Range} = \left\{ \left\langle \sqrt{4-t^2},\ \tfrac{1}{t-1} \right\rangle \;:\; t \in [-2, 1) \cup (1, 2] \right\}$$

Check: $t = 0$ gives $\langle 2, -1\rangle$; $t = 2$ gives $\langle 0, 1\rangle$; $t = -2$ gives $\langle 0, -\tfrac{1}{3}\rangle$; $t = 1$ is not allowed.

CONCEPT

$t^2 \leq a^2 \iff -a \leq t \leq a$ (for $a > 0$).

Here $t^2 \leq 4$ gives $-2 \leq t \leq 2$.

CONCEPT

Remove the excluded point from the interval: $[-2, 2]$ without $t = 1$ splits into $[-2, 1)$ and $(1, 2]$.

Answer: Domain $= [-2, 1) \cup (1, 2]$; Range $= \{\langle \sqrt{4-t^2}, \tfrac{1}{t-1}\rangle : t \in [-2, 1) \cup (1, 2]\}$.

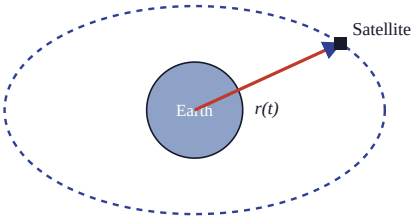
EXERCISE 11.1

Q1 **Give two real-life examples where vector-valued functions are necessary.**

ہدایت: ایسی دو مثالیں لکھیں جن میں کوئی چیز وقت t کے ساتھ بدلتی ہو اور اسے بیان کرنے کے لیے ایک سے زیادہ components (مقدار اور سمت دونوں) چاہئیں۔ ہر مثال کے ساتھ اس کا vector function اور مختصر وجہ لکھ دیں۔

Solution:

Example 1: The position of a satellite orbiting the earth.



At every time t the satellite is at a point in space, so three coordinates $x(t), y(t), z(t)$ are needed.

Combine them into one vector:

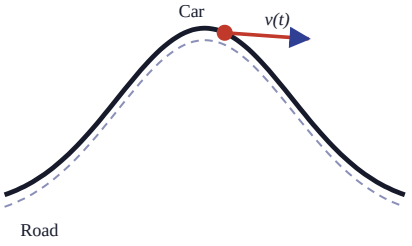
$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

One real number cannot describe a position in space, so a vector-valued function is necessary.

CONCEPT

Use a vector function when the quantity has **several components** that change with t (position in 2D / 3D).

Example 2: The velocity of a car moving along a curved road.



On a curved road the car's **direction** changes continuously, and its speed may change too.

Velocity has magnitude *and* direction, so it is written with components:

$$\vec{v}(t) = v_x(t)\,\hat{i} + v_y(t)\,\hat{j}$$

A single scalar cannot show both speed and direction, so a vector-valued function is necessary.

CONCEPT

Velocity is tangent to the path and is a **vector**: magnitude (speed) + direction.

Q2 Construct a vector function for a particle moving in a circle.

ہدایت: مرکز $O(0,0)$ اور radius a والے دائرے کی parametric equations $x = a\cos t,\ y = a\sin t$ لکھیں. پھر x کو $\hat{i}$ کے ساتھ اور y کو $\hat{j}$ کے ساتھ جوڑ کر $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ بنا لیں.

Solution:

Parametric equations. A circle with centre $O(0,0)$ and radius $a\ (>0)$ has

$$x = a\cos t, \qquad y = a\sin t$$

Check. Squaring and adding:

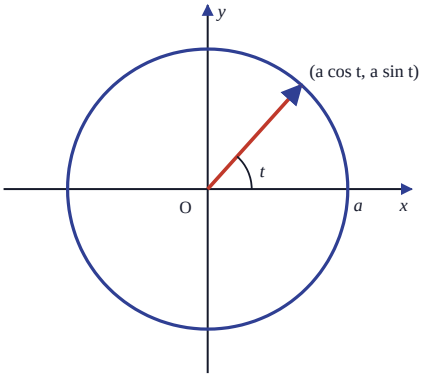
$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2(1) = a^2$$

which is the equation of the circle.

Vector function.

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}, \qquad 0 \le t \le 2\pi$$



CONCEPT

Parametric form of a circle: $x = a\cos t,\ y = a\sin t$ uses $\cos^2 t + \sin^2 t = 1$.

Q3 Write a vector function for a car moving along a parabolic path.

ہدایت: parabola $x^2 = 4ay$ کی parametric equations $x = 2at,\ y = at^2$ ہوتی ہیں. انہیں $\hat{i}$ اور $\hat{j}$ کے ساتھ جوڑ کر $\vec{r}(t)$ لکھیں.

Solution:

Parametric equations. For the parabola $x^2 = 4ay$ take

$$x = 2at, \qquad y = at^2, \qquad a \in \mathbb{R},\ a \neq 0$$

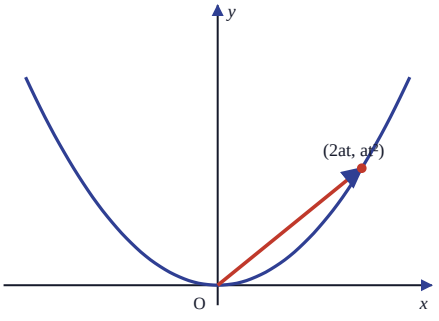
Check.

$$x^2 = (2at)^2 = 4a^2t^2 = 4a\,(at^2) = 4a\,y \ \checkmark$$

Vector function.

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\vec{r}(t) = 2at\,\hat{i} + at^2\,\hat{j}$$



CONCEPT

Parabola $x^2 = 4ay$ has parametric form $(2at,\ at^2)$: put $x = 2at$, then $y = \dfrac{x^2}{4a} = at^2$.

Q4 Create a 3D vector function for a drone flying upward in a spiral.

ہدایت: spiral میں x اور y دائرے کی طرح گھومتے ہیں $(a\cos t,\ a\sin t)$ اور ساتھ ساتھ z سیدھا اوپر بڑھتا ہے $(z = bt)$.
تینوں components کو $\hat{i}, \hat{j}, \hat{k}$ کے ساتھ لکھ دیں.

Solution:

Components. The drone moves in a circle in the xy-plane while rising steadily:

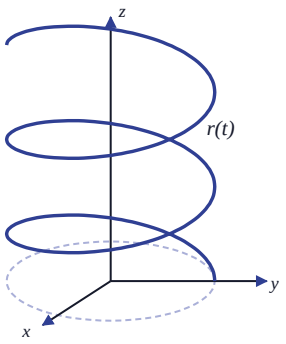
$$x = a\cos t, \qquad y = a\sin t, \qquad z = bt$$

where $a > 0$ is the radius of the spiral and $b > 0$ is the vertical speed.

Vector function.

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$$



CONCEPT

Helix = circle in xy-plane + steady motion along z.

Q5 Given $x(t) = t^2 + 1,\ y(t) = \sqrt{t}$, construct $\vec{r}(t)$.

ہدایت: دونوں دیے گئے functions کو سیدھا vector form میں لکھنا ہے: $x(t)$ کو $\hat{i}$ کے ساتھ اور $y(t)$ کو $\hat{j}$ کے ساتھ لگائیں.

Solution:

Given parametric equations: $x(t) = t^2 + 1,\ y(t) = \sqrt{t}$.

Two components, so

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

Substitute the given functions:

$$\vec{r}(t) = (t^2 + 1)\,\hat{i} + \sqrt{t}\,\hat{j}, \qquad t \geq 0$$

CONCEPT

2 components $\Rightarrow\ \vec{r} = x\hat{i} + y\hat{j}$.
3 components $\Rightarrow\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
$\sqrt{t}$ needs $t \geq 0$.

Q6 **Given $x(t) = \dfrac{1}{t+4},\ y(t) = \ln\sqrt{t},\ z(t) = \sec^{-1}(t)$, construct $\vec{r}(t)$.**

ہدایت: یہاں تین components ہیں، اس لیے $\hat{i}, \hat{j}, \hat{k}$ تینوں استعمال ہوں گے: $x \to \hat{i},\ y \to \hat{j},\ z \to \hat{k}$.

Solution:

Given: $x(t) = \dfrac{1}{t+4},\ \ y(t) = \ln\sqrt{t},\ \ z(t) = \sec^{-1}(t).$

Three components, so $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$.

Substitute:

$$\vec{r}(t) = \frac{1}{t+4}\,\hat{i} + \ln\sqrt{t}\,\hat{j} + \sec^{-1}(t)\,\hat{k}$$

CONCEPT

Log law: $\ln\sqrt{t} = \ln t^{1/2} = \frac{1}{2}\ln t$ (so either form is correct).
Domain: $t > 0$ for $\ln$, and $|t| \geq 1$ for $\sec^{-1} t$.

Q7 **Find the domain and range of the following vector-valued functions.**
(i) $\vec{r}(t) = \sqrt{t}\,\hat{i} + \ln t\,\hat{j}$ **(ii)** $\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j}$
(iii) $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + e^t\,\hat{k}$ **(iv)** $\vec{r}(t) = \sin(t)\,\hat{j} + \cos(t)\,\hat{i} + t\,\hat{k}$ **(see note on (iv))**

ہدایت: (1) ہر component کا الگ domain نکالیں: جڑ کے اندر ≥ 0، $\ln$ کے اندر > 0. (2) سب domains کا intersection لیں، یہی $\vec{r}(t)$ کا domain ہے۔ (3) Range domain میں ہر t کے لیے بننے والے تمام vectors $\langle x(t), y(t), \ldots\rangle$ لکھیں۔

Solution:

(i) $\vec{r}(t) = \sqrt{t}\,\hat{i} + \ln t\,\hat{j}$

Domain of each component.

$x(t) = \sqrt{t}$ is defined when $t \geq 0$. $\quad y(t) = \ln t$ is defined when $t > 0$.
$D_x = [0, \infty)$ $\quad D_y = (0, \infty)$

Intersection.

$$\text{Domain} = D_x \cap D_y = [0,\infty) \cap (0,\infty) = (0,\infty)$$

Range.

$$\text{Range} = \{\langle x(t), y(t)\rangle \mid t \in (0,\infty)\} = \{\langle \sqrt{t}, \ln t\rangle \mid t \in (0,\infty)\}$$

CONCEPT

$\ln t$ is defined only for $t > 0$ (not $t = 0$), so the point $t = 0$ allowed by $\sqrt{t}$ is removed by the intersection.

(ii) $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$x(t) = \cos t$ and $y(t) = \sin t$ are defined for all $t \in \mathbb{R}$.
$D_x = (-\infty, \infty),\ D_y = (-\infty, \infty)$, so

$$\text{Domain} = (-\infty, \infty)$$

$$\begin{aligned}\text{Range} &= \{\langle x(t), y(t)\rangle \mid t \in (-\infty,\infty)\} \\ &= \{\langle \cos t, \sin t\rangle \mid t \in (-\infty,\infty)\}\end{aligned}$$

CONCEPT

Since $\cos^2 t + \sin^2 t = 1$, these vectors are exactly the points of the unit circle $x^2 + y^2 = 1$.

(iii) $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + e^t\,\hat{k}$

$x(t) = t,\ y(t) = t^2$ and $z(t) = e^t$ are defined for all $t \in \mathbb{R}$.

$D_x = D_y = D_z = (-\infty, \infty)$, so

$$\text{Domain} = (-\infty, \infty)$$

$$\begin{aligned}\text{Range} &= \{\langle x(t), y(t), z(t)\rangle \mid t \in (-\infty, \infty)\} \\ &= \{\langle t, t^2, e^t\rangle \mid t \in (-\infty, \infty)\}\end{aligned}$$

> **CONCEPT**
> Polynomials ($t,\ t^2$) and the exponential e^t are defined for every real t.

(iv) $\vec{r}(t) = \sin t\,\hat{j} + \cos t\,\hat{i} + t\,\hat{k}$

Here $x(t) = \cos t$ (the $\hat{i}$ term), $y(t) = \sin t$ (the $\hat{j}$ term) and $z(t) = t$ (the $\hat{k}$ term); all are defined for every real t.

$D_x = D_y = D_z = (-\infty, \infty)$, so

$$\text{Domain} = (-\infty, \infty)$$

$$\text{Range} = \{\langle \cos t, \sin t, t\rangle \mid t \in (-\infty, \infty)\}$$

> **CONCEPT**
> Always match each term with its unit vector: coefficient of $\hat{i}$ is x, of $\hat{j}$ is y, of $\hat{k}$ is z.

Note: The question prints the $\hat{j}$-term first ($\sin t\,\hat{j} + \cos t\,\hat{i} + t\,\hat{k}$), so the components are read from the unit vectors: $x = \cos t,\ y = \sin t,\ z = t$. The domain is the same in either order.

Q8 **Determine which of the following functions are scalar functions or vector-valued functions. Give appropriate reasoning to support the answer.**

(i) $f(t) = (3t^2 - 5)^2$ **(ii)** $h(t) = \langle t^2, e^t, \ln\sqrt{t}\rangle$

(iii) $g(t) = \tan(t) + \csc(t)$ **(iv)** $j(t) = \left\langle \dfrac{1}{t^2+1}, \dfrac{1}{t^2+1} \right\rangle$

ہدایت: دیکھیں کہ function کا output ایک اکیلا عدد ہے یا $\langle\ ,\ \rangle$ میں لکھے کئی components. ایک output ⇒ scalar function؛ دو یا زیادہ components ⇒ vector-valued function. وجہ (reason) ضرور لکھیں۔

Solution:

(i) $f(t) = (3t^2 - 5)^2$ — **Scalar function.**
Reason: for each t the function gives only **one** real number as output.

(ii) $h(t) = \langle t^2, e^t, \ln\sqrt{t}\rangle$ — **Vector-valued function.**
Reason: the function has **three** components $t^2,\ e^t,\ \ln\sqrt{t}$, so the output is a vector.

(iii) $g(t) = \tan t + \csc t$ — **Scalar function.**
Reason: $\tan t$ and $\csc t$ are added to give **one** real number as output (the "+" sign does not make a vector).

(iv) $j(t) = \left\langle \dfrac{1}{t^2+1}, \dfrac{1}{t^2+1} \right\rangle$ — **Vector-valued function.**
Reason: the function has **two** components, so the output is a vector.

> **CONCEPT**
> **Count the outputs.**
> One value ⟶ scalar function.
> $\langle a, b\rangle$ or $\langle a, b, c\rangle$ ⟶ vector function.
> Adding two scalars (e.g. $\tan t + \csc t$) still gives a scalar.

Derivative of a Vector-Valued Function

A vector-valued function of a single variable maps a scalar (usually time t) to a vector in $\mathbb{R}^n$. In three-dimensional space it is written as

$$\vec{r}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

where f_1, f_2, f_3 are scalar functions of t (the components). The derivative with respect to t is defined as

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

If the limit exists, $\vec{r}\,'(t)$ is found by differentiating each component:

$$\vec{r}\,'(t) = \langle f_1'(t), f_2'(t), f_3'(t) \rangle = f_1'(t)\hat{i} + f_2'(t)\hat{j} + f_3'(t)\hat{k}$$

CONCEPT

Differentiate **component by component**; the unit vectors $\hat{i}, \hat{j}, \hat{k}$ are constants.

EXERCISE 11.2

Q1 **Find the first order derivatives of the following vector valued functions with respect to t.**

ہدایت: ہر component کو الگ الگ t کے respect میں differentiate کریں، پھر جوابات کو دوبارہ $\hat{i}, \hat{j}, \hat{k}$ کے ساتھ جوڑ دیں۔ $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

(i) $\vec{r}(t) = (t^2+1)\hat{i} + (t^3+t^2+3t)\hat{j}$

Differentiate each component separately.

$$\frac{d}{dt}(t^2+1) = 2t, \qquad \frac{d}{dt}(t^3+t^2+3t) = 3t^2+2t+3$$

Combine.

$$\vec{r}\,'(t) = 2t\,\hat{i} + (3t^2+2t+3)\,\hat{j}$$

POWER RULE

$\dfrac{d}{dt}t^n = nt^{n-1}$, derivative of a constant is 0.

(ii) $\vec{r}(t) = (t\sin t - \cos t)\hat{i} + (\cot t + \ln t)\hat{j} + \ln(1+t^2)\hat{k}$

$\hat{i}$ component.

$$\begin{aligned}\frac{d}{dt}(t\sin t - \cos t) &= \frac{d}{dt}(t\sin t) - \frac{d}{dt}(\cos t) \\ &= (\sin t + t\cos t) + \sin t \\ &= 2\sin t + t\cos t\end{aligned}$$

PRODUCT RULE

$(uv)' = u'v + uv'$

$\dfrac{d}{dt}(t\sin t) = \sin t + t\cos t$

$\hat{j}$ component.

$$\frac{d}{dt}(\cot t + \ln t) = -\csc^2 t + \frac{1}{t}$$

FORMULAS

$(\cot t)' = -\csc^2 t,$

$(\ln t)' = \dfrac{1}{t}$

$\hat{k}$ component.

$$\frac{d}{dt}\ln(1+t^2) = \frac{1}{1+t^2}\cdot(2t) = \frac{2t}{1+t^2}$$

Combine.

$$\vec{r}\,'(t) = (2\sin t + t\cos t)\hat{i} + \left(\frac{1}{t} - \csc^2 t\right)\hat{j} + \frac{2t}{1+t^2}\,\hat{k}$$

CHAIN RULE

$$\frac{d}{dt}\ln u = \frac{u'}{u}$$

(iii) $\vec{r}(t) = \dfrac{2}{\sqrt{1-t^2}}\hat{i} + \dfrac{\sqrt{3}}{1+t^2}\hat{k}$

$\hat{i}$ component. Write $\dfrac{2}{\sqrt{1-t^2}} = 2(1-t^2)^{-1/2}$.

$$\frac{d}{dt}2(1-t^2)^{-1/2} = 2\Big\{-\tfrac{1}{2}(1-t^2)^{-3/2}(-2t)\Big\} = \frac{2t}{(1-t^2)^{3/2}}$$

$\hat{k}$ component. Write $\dfrac{\sqrt{3}}{1+t^2} = \sqrt{3}(1+t^2)^{-1}$.

$$\frac{d}{dt}\sqrt{3}(1+t^2)^{-1} = \sqrt{3}\{(-1)(1+t^2)^{-2}(2t)\} = -\frac{2\sqrt{3}\,t}{(1+t^2)^2}$$

Combine.

$$\vec{r}\,'(t) = \frac{2t}{(1-t^2)^{3/2}}\hat{i} - \frac{2\sqrt{3}\,t}{(1+t^2)^2}\,\hat{k}$$

CHAIN + POWER RULE

$\dfrac{d}{dt}u^n = n\,u^{n-1}u'$. Rewrite roots and fractions as powers first.

Q2 **Find the first order derivatives of the following vector valued functions with respect to t at the specified values of t.**

ہدایت: پہلے ہر component کا derivative نکالیں، آخر میں دی گئی t کی value ڈالیں. اگر کسی component (یا خود derivative) اس t پر define نہ ہو تو وہاں $\vec{r}\,'(t)$ exist نہیں کرتا.

(i) $\vec{r}(t) = \left(\dfrac{t^2+1}{\sin t}\right)\hat{i} + \left(\sqrt{t}\,e^t\right)\hat{j}$ **at** $t = \dfrac{\pi}{2}$

$\hat{i}$ component (quotient rule).

$$\frac{d}{dt}\frac{t^2+1}{\sin t} = \frac{(\sin t)(2t) - (t^2+1)\cos t}{\sin^2 t} = \frac{2t\sin t - (t^2+1)\cos t}{\sin^2 t}$$

QUOTIENT RULE

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$\hat{j}$ component (product rule).

$$\frac{d}{dt}\left(\sqrt{t}\,e^{t}\right) = \sqrt{t}\,e^{t} + e^{t}\cdot\frac{1}{2\sqrt{t}} = \sqrt{t}\,e^{t} + \frac{e^{t}}{2\sqrt{t}}$$

Combine.

$$\vec{r}\,'(t) = \frac{2t\sin t - (t^{2}+1)\cos t}{\sin^{2} t}\,\hat{i} + \left(\sqrt{t}\,e^{t} + \frac{e^{t}}{2\sqrt{t}}\right)\hat{j}$$

Put $t = \pi/2$ ($\sin\frac{\pi}{2} = 1,\ \cos\frac{\pi}{2} = 0$).

$$\hat{i}\text{-part} = \frac{2\cdot\frac{\pi}{2}\cdot 1 - \left(\left(\frac{\pi}{2}\right)^{2}+1\right)\cdot 0}{1^{2}} = \pi$$

$$\hat{j}\text{-part} = \sqrt{\tfrac{\pi}{2}}\,e^{\pi/2} + \frac{e^{\pi/2}}{2\sqrt{\pi/2}} = e^{\pi/2}\left(\sqrt{\tfrac{\pi}{2}} + \frac{1}{2\sqrt{\pi/2}}\right) \approx 7.95$$

$$\vec{r}\,'\!\left(\frac{\pi}{2}\right) = \pi\,\hat{i} + e^{\pi/2}\left(\sqrt{\tfrac{\pi}{2}} + \frac{1}{2\sqrt{\pi/2}}\right)\hat{j} \;\approx\; 3.14\,\hat{i} + 7.95\,\hat{j}$$

(ii) $\vec{r}(t) = \left(t\sqrt{t+2}\right)\hat{i} + \ln\left(t + \sqrt{t^{2}+1}\right)\hat{j}$ **at** $t = 0$

$\hat{i}$ component.

$$\frac{d}{dt}\left(t\sqrt{t+2}\right) = \sqrt{t+2}\,(1) + t\cdot\frac{1}{2\sqrt{t+2}} = \sqrt{t+2} + \frac{t}{2\sqrt{t+2}}$$

$\hat{j}$ component.

$$\begin{aligned}\frac{d}{dt}\ln\left(t+\sqrt{t^{2}+1}\right) &= \frac{1}{t+\sqrt{t^{2}+1}}\left(1 + \frac{2t}{2\sqrt{t^{2}+1}}\right)\\ &= \frac{1}{t+\sqrt{t^{2}+1}}\cdot\frac{\sqrt{t^{2}+1}+t}{\sqrt{t^{2}+1}}\\ &= \frac{1}{\sqrt{t^{2}+1}}\end{aligned}$$

Combine, then $t = 0$.

$$\vec{r}\,'(t) = \left(\sqrt{t+2} + \frac{t}{2\sqrt{t+2}}\right)\hat{i} + \frac{1}{\sqrt{t^{2}+1}}\,\hat{j}$$

$$\vec{r}\,'(0) = \sqrt{2}\,\hat{i} + \frac{1}{1}\,\hat{j}$$

$$\vec{r}\,'(0) = \sqrt{2}\,\hat{i} + \hat{j}$$

CONCEPT

Substitute the value of t **after** differentiating. $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$.

CHAIN RULE

$\frac{d}{dt}\sqrt{u} = \frac{u'}{2\sqrt{u}}$.

Simplify $\frac{1}{A}\cdot\frac{A}{B} = \frac{1}{B}$ where $A = t + \sqrt{t^{2}+1}$.

(iii) $\vec{r}(t) = \arctan(t)\hat{i} + \left(\dfrac{\sin(t^2)}{t}\right)\hat{j} + \sqrt[3]{1+t}\,\hat{k}$ **at** $t = 0$

$\hat{i}$ component. $\dfrac{d}{dt}\tan^{-1} t = \dfrac{1}{1+t^2}$.

$\hat{j}$ component (quotient + chain rule).

$$\frac{d}{dt}\frac{\sin(t^2)}{t} = \frac{t\cos(t^2)(2t) - \sin(t^2)}{t^2} = \frac{2t^2\cos(t^2) - \sin(t^2)}{t^2}$$

$\hat{k}$ component. $\dfrac{d}{dt}(1+t)^{1/3} = \dfrac{1}{3}(1+t)^{-2/3}$.

Combine.

$$\vec{r}\,'(t) = \frac{1}{1+t^2}\hat{i} + \frac{2t^2\cos(t^2) - \sin(t^2)}{t^2}\hat{j} + \frac{1}{3}(1+t)^{-2/3}\hat{k}$$

Check $t = 0$. The $\hat{j}$-component $\dfrac{\sin(t^2)}{t}$ of $\vec{r}(t)$, and the $\hat{j}$-component of $\vec{r}\,'(t)$, both have t in the denominator and are **not defined at** $t = 0$.

Therefore $\vec{r}\,'(0)$ does not exist.

EXISTENCE

$\vec{r}\,'(t_0)$ exists only if **every** component is defined (and differentiable) at t_0.

One undefined component $\Rightarrow$ no derivative.

(iv) $\vec{r}(t) = t\,e^{-t}\hat{i} + \cos(3t)\hat{j} + \ln(1+t^2)\hat{k}$ **at** $t = 0$

$\hat{i}$ component (product rule).

$$\frac{d}{dt}(te^{-t}) = e^{-t}(1) + t\,e^{-t}(-1) = e^{-t} - te^{-t} = e^{-t}(1-t)$$

$\hat{j}$ component. $\dfrac{d}{dt}\cos 3t = -3\sin 3t$.

$\hat{k}$ component. $\dfrac{d}{dt}\ln(1+t^2) = \dfrac{2t}{1+t^2}$.

Combine, then $t = 0$.

$$\vec{r}\,'(t) = e^{-t}(1-t)\hat{i} - 3\sin 3t\,\hat{j} + \frac{2t}{1+t^2}\hat{k}$$

$$\vec{r}\,'(0) = e^{0}(1)\hat{i} - 3\sin 0\,\hat{j} + \frac{0}{1}\hat{k} = 1\hat{i} - 0\,\hat{j} + 0\,\hat{k}$$

$$\vec{r}\,'(0) = \hat{i}$$

CHAIN RULE

$\dfrac{d}{dt}\cos(kt) = -k\sin(kt)$,

$\dfrac{d}{dt}e^{-t} = -e^{-t}$.

EXERCISE 11.2 — MULTIPLE CHOICE QUESTIONS

ہدایت: ہر MCQ میں پہلے $\vec{r}'(t)$ (یا $\vec{r}''(t)$) نکالیں، پھر دی گئی value ڈالیں اور options سے ملائیں. نیچے ہر سوال کا جواب مختصر وجہ کے ساتھ دیا گیا ہے.

CONCEPT

Velocity $\vec{v} = \vec{r}'$.
Acceleration $\vec{a} = \vec{v}' = \vec{r}''$.
$|t|$ is not differentiable at $t = 0$ (corner).

1. The derivative of a vector-valued function $\vec{r}(t)$ gives: A) Change of position B) Only direction C) Same function D) Second derivative
Answer: A. $\vec{r}'(t)$ is the rate of change of position (velocity).

2. If $\vec{r}(t) = \langle t^2, t^3 \rangle$, then $\vec{v}(t) =$ A) $\langle 2t, 3t^2 \rangle$ B) $\langle t, 3t^2 \rangle$ C) $\langle 2, 3 \rangle$ D) $\langle t^2, t^3 \rangle$
Answer: A. $\vec{v} = \vec{r}' = \langle 2t, 3t^2 \rangle$.

3. The acceleration vector is obtained by: A) Integrating velocity B) 1st derivative of position C) 2nd derivative of position D) Time $\times$ position
Answer: C. $\vec{a} = \vec{v}' = \vec{r}''$.

4. If $\vec{r}(t) = \langle at^2, bt^2, ct^2 \rangle$, then $\vec{r}'(1) =$ A) $\langle 2a, 2b, 2c \rangle$ B) $\langle a, b, 2c \rangle$ C) $\langle 2, 2, 2 \rangle$ D) $\langle a^2, b^2, c^2 \rangle$
Answer: A. $\vec{r}'(t) = \langle 2at, 2bt, 2ct \rangle$; at $t = 1$: $\langle 2a, 2b, 2c \rangle$.

5. If $\vec{r}(t) = \langle e^t, e^t, e^t \rangle$, then $\vec{r}'(t) - \vec{r}(t) =$ A) $\langle 0, 0, 0 \rangle$ B) $\langle 1, 1, 1 \rangle$ C) $\langle e^t, e^t, e^t \rangle$ D) Undefined
Answer: A. $\vec{r}' = \langle e^t, e^t, e^t \rangle = \vec{r}$, so the difference is $\mathbf{0}$.

6. If $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, then $\vec{r}'(\pi/2) =$ A) $\langle -1, 0, 1 \rangle$ B) $\langle 0, 1, 1 \rangle$ C) $\langle 1, 0, 1 \rangle$ D) $\langle 0, -1, 1 \rangle$
Answer: A. $\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$; at $\frac{\pi}{2}$: $\langle -1, 0, 1 \rangle$.

7. If $\vec{r}(t) = \langle t^2, 2t, 5 \rangle$ and $\vec{s}(t) = \langle t^2, 2t, 5 + t \rangle$: A) $\vec{r}' = \vec{s}'$ B) $\vec{r}' \neq \vec{s}'$ C) Both identical D) Neither differentiable
Answer: B. $\vec{r}' = \langle 2t, 2, 0 \rangle$ but $\vec{s}' = \langle 2t, 2, 1 \rangle$.

8. If $\vec{r}(t) = \langle t^2, |t|, t^3 \rangle$, at which point is $\vec{r}'(t)$ not defined? A) $t = 1$ B) $t = 0$ C) $t = -1$ D) $t = 2$
Answer: B. $|t|$ has a sharp corner at $t = 0$, so it is not differentiable there.

Q	1	2	3	4	5	6	7	8
Ans	A	A	C	A	A	A	B	B

Velocity and Acceleration

If $\vec{r}$ is the position vector of a particle moving along a smooth curve in space, then $\frac{d}{dt}\vec{r}(t)$ represents the **velocity vector** of the particle, which is tangent to the curve, i.e. $\frac{d}{dt}\vec{r}(t) = \vec{v}(t)$. The derivative of the velocity vector with respect to time t is the **acceleration** of the particle:

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t)$$

Also, **speed** $= |\vec{v}(t)|$ and the **direction of motion** is the unit vector $\hat{v}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$.

FORMULAS

$\vec{v} = \vec{r}',\ \vec{a} = \vec{v}'$

Speed $= |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Direction $\hat{v} = \vec{v}/|\vec{v}|$

$\vec{v} \cdot \vec{a} > 0$: speeding up; < 0: slowing down; $= 0$: speed momentarily constant.

EXERCISE 11.3

In Q1–Q5, each problem involves a vector-valued function representing the particle's position in the xy-plane or in 3D space. Find the velocity, acceleration, speed, and direction of motion at a specified value of t. Interpret the results by describing the particle's motion at that instant, including how fast and in what direction it is moving, whether it is speeding up or slowing down, and how its path is curving.

ہدایت (Q1-Q5): ترتیب یہ ہے: (a) $\vec{v} = \frac{d\vec{r}}{dt}$ نکالیں، پھر دیا گیا t رکھیں. (b) $\vec{a} = \frac{d\vec{v}}{dt}$ نکالیں، پھر t رکھیں. (c) Speed $= |\vec{v}|$ نکالیں. (d) سمت کے لیے $\hat{v} = \vec{v}/|\vec{v}|$ لکھیں. (e) آخر میں $\vec{v} \cdot \vec{a}$ دیکھ کر بتائیں کہ particle تیز ہو رہا ہے یا آہستہ، اور motion کی تشریح لکھیں.

Q1 $\vec{r}(t) = 2\sin t\,\hat{i} + 3t\,\hat{j}$ at $t = \dfrac{\pi}{2}$

CONCEPT

$\frac{d}{dt}\sin t = \cos t;\ \cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1.$

(a) Velocity.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(2\sin t\,\hat{i} + 3t\,\hat{j}) = 2\cos t\,\hat{i} + 3\,\hat{j}$$
$$\vec{v}\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2}\,\hat{i} + 3\hat{j} = 0\,\hat{i} + 3\,\hat{j} = 3\,\hat{j}$$

CONCEPT

$\vec{v} \cdot \vec{a} = 0$ means acceleration is perpendicular to velocity: it only **turns** the path, speed is not changing at that instant.

(b) Acceleration.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(2\cos t\,\hat{i} + 3\hat{j}) = -2\sin t\,\hat{i} + 0\,\hat{j}$$
$$\vec{a}\left(\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2}\,\hat{i} = -2(1)\hat{i} = -2\,\hat{i}$$

(c) Speed.

$$|\vec{v}(t)| = \sqrt{(2\cos t)^2 + 3^2} \ \Rightarrow\ \left|\vec{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{0^2 + 3^2} = 3 \text{ units/s}$$

(d) Direction of motion.

$$\hat{v}\left(\frac{\pi}{2}\right) = \frac{\vec{v}(\pi/2)}{|\vec{v}(\pi/2)|} = \frac{0\,\hat{i} + 3\,\hat{j}}{3} = \hat{j}$$

(e) Speeding up or slowing down? $\vec{v} \cdot \vec{a} = (0)(-2) + (3)(0) = 0$.

Interpretation: At $t = \frac{\pi}{2}$ the particle moves vertically upward ($+y$) with a speed of 3 units per second. The acceleration $-2\,\hat{i}$ acts in the negative x-direction, perpendicular to the velocity, so the speed is momentarily constant and the path bends to the left (sinusoidal motion in the x-component).

Q2 $\vec{r}(t) = e^t\,\hat{i} + e^{-t}\,\hat{j}$ at $t = 0$

> **CONCEPT**
> $\frac{d}{dt}e^{-t} = -e^{-t}$ (chain rule); $e^0 = 1$.

(a) Velocity.

$$\vec{v}(t) = \frac{d}{dt}(e^t\hat{i} + e^{-t}\hat{j}) = e^t\hat{i} + e^{-t}(-1)\hat{j} = e^t\hat{i} - e^{-t}\hat{j}$$

$$\vec{v}(0) = e^0\hat{i} - e^0\hat{j} = \hat{i} - \hat{j}$$

(b) Acceleration.

$$\vec{a}(t) = \frac{d}{dt}(e^t\hat{i} - e^{-t}\hat{j}) = e^t\hat{i} - e^{-t}(-1)\hat{j} = e^t\hat{i} + e^{-t}\hat{j}$$

$$\vec{a}(0) = e^0\hat{i} + e^0\hat{j} = \hat{i} + \hat{j}$$

(c) Speed.

$$|\vec{v}(t)| = \sqrt{(e^t)^2 + (-e^{-t})^2} = \sqrt{e^{2t} + e^{-2t}} \;\Rightarrow\; |\vec{v}(0)| = \sqrt{1+1} = \sqrt{2}\ \text{units/s}$$

(d) Direction of motion.

$$\hat{v}(0) = \frac{\hat{i} - \hat{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

(e) $\vec{v}\cdot\vec{a} = (1)(1) + (-1)(1) = 0$.

Interpretation: The particle moves at 45° downward and to the right with a speed of $\sqrt{2}$ units per second. The acceleration $\hat{i} + \hat{j}$ is perpendicular to the velocity, so the speed is momentarily constant while the path curves (the x-component grows and the y-component decays exponentially).

Q3 $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 2t\,\hat{k}$ at $t = \pi$

> **CONCEPT**
> $\sin^2 t + \cos^2 t = 1$ makes the speed of a helix **constant**. $\sin\pi = 0,\ \cos\pi = -1$.

(a) Velocity.

$$\vec{v}(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 2\,\hat{k}$$

$$\vec{v}(\pi) = -\sin\pi\,\hat{i} + \cos\pi\,\hat{j} + 2\hat{k} = 0\,\hat{i} - 1\,\hat{j} + 2\,\hat{k}$$

(b) Acceleration.

$$\vec{a}(t) = \frac{d}{dt}(-\sin t\,\hat{i} + \cos t\,\hat{j} + 2\hat{k}) = -\cos t\,\hat{i} - \sin t\,\hat{j} + 0\,\hat{k}$$

$$\vec{a}(\pi) = -\cos\pi\,\hat{i} - \sin\pi\,\hat{j} = -(-1)\hat{i} - 0\,\hat{j} = \hat{i}$$

(c) Speed.

$$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{1+4} = \sqrt{5} \;\Rightarrow\; |\vec{v}(\pi)| = \sqrt{5}\ \text{units/s}$$

(d) Direction of motion.

$$\hat{v}(\pi) = \frac{0\,\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{5}} = 0\,\hat{i} - \frac{1}{\sqrt{5}}\hat{j} + \frac{2}{\sqrt{5}}\hat{k}$$

(e) $\vec{v}\cdot\vec{a} = (0)(1) + (-1)(0) + (2)(0) = 0$.

Interpretation: The particle follows a helical path, moving in the negative y-direction and upward in z with speed $\sqrt{5}$ units per second. The acceleration $\hat{i}$ points toward the axis of the helix (circular motion in the xy-plane); it is

perpendicular to the velocity, so the speed stays constant.

Q4 $\vec{r}(t) = t^2\,\hat{i} + t\,\hat{j} + t^3\,\hat{k}$ **at** $t = 2$

CONCEPT

$\vec{v}\cdot\vec{a} > 0 \Rightarrow$ the particle is **speeding up**.

Power rule: $(t^n)' = nt^{n-1}$.

(a) Velocity.

$$\vec{v}(t) = \frac{d}{dt}(t^2\hat{i} + t\hat{j} + t^3\hat{k}) = 2t\,\hat{i} + 1\,\hat{j} + 3t^2\,\hat{k}$$

$$\vec{v}(2) = 2(2)\hat{i} + \hat{j} + 3(2)^2\hat{k} = 4\hat{i} + \hat{j} + 12\hat{k}$$

(b) Acceleration.

$$\vec{a}(t) = \frac{d}{dt}(2t\hat{i} + \hat{j} + 3t^2\hat{k}) = 2\,\hat{i} + 0\,\hat{j} + 6t\,\hat{k}$$

$$\vec{a}(2) = 2\hat{i} + 0\,\hat{j} + 6(2)\hat{k} = 2\hat{i} + 12\hat{k}$$

(c) Speed.

$$|\vec{v}(t)| = \sqrt{(2t)^2 + 1^2 + (3t^2)^2} = \sqrt{4t^2 + 1 + 9t^4}$$

$$|\vec{v}(2)| = \sqrt{4(2)^2 + 1 + 9(2)^4} = \sqrt{16 + 1 + 144} = \sqrt{161} \approx 12.69 \text{ units/s}$$

(d) Direction of motion.

$$\hat{v}(2) = \frac{4\hat{i} + \hat{j} + 12\hat{k}}{\sqrt{161}}$$

(e) $\vec{v}\cdot\vec{a} = (4)(2) + (1)(0) + (12)(12) = 8 + 144 = 152 > 0$.

Interpretation: At $t = 2$ the particle moves with speed $\sqrt{161} \approx 12.7$ units per second, mostly in the z-direction. Since $\vec{v}\cdot\vec{a} = 152 > 0$ it is speeding up, with velocity increasing in the x- and z-directions, consistent with the polynomial nature of the path.

Q5 $\vec{r}(t) = e^t \sin t\,\hat{i} + (t^3 - 2t)\,\hat{j} + \arctan(2t)\,\hat{k}$ **at** $t = 0$

FORMULAS

Product rule for $e^t \sin t$.

$$\frac{d}{dt}\tan^{-1} u = \frac{u'}{1 + u^2}$$

(a) Velocity. Differentiate each component.

$$\frac{d}{dt}(e^t \sin t) = e^t \sin t + e^t \cos t = e^t(\cos t + \sin t)$$

$$\frac{d}{dt}(t^3 - 2t) = 3t^2 - 2, \qquad \frac{d}{dt}\tan^{-1}(2t) = \frac{1}{1 + (2t)^2}\cdot 2 = \frac{2}{1 + 4t^2}$$

$$\vec{v}(t) = e^t(\cos t + \sin t)\hat{i} + (3t^2 - 2)\hat{j} + \frac{2}{1 + 4t^2}\hat{k}$$

$$\vec{v}(0) = e^0(\cos 0 + \sin 0)\hat{i} + (0 - 2)\hat{j} + \frac{2}{1 + 0}\hat{k} = 1\,\hat{i} - 2\,\hat{j} + 2\,\hat{k}$$

(b) Acceleration. Differentiate $\vec{v}(t)$ again.

$$\frac{d}{dt}\left[e^t(\cos t+\sin t)\right] = e^t(\cos t+\sin t)+e^t(-\sin t+\cos t)$$
$$= 2e^t\cos t$$
$$\frac{d}{dt}(3t^2-2) = 6t$$
$$\frac{d}{dt}2(1+4t^2)^{-1} = -2(1+4t^2)^{-2}(8t) = -\frac{16t}{(1+4t^2)^2}$$
$$\vec{a}(t) = 2e^t\cos t\,\hat{i}+6t\,\hat{j}-\frac{16t}{(1+4t^2)^2}\hat{k}$$
$$\vec{a}(0) = 2(1)(1)\hat{i}+0\,\hat{j}-0\,\hat{k} = 2\,\hat{i}$$

CONCEPT

Acceleration = derivative of the **velocity** (not of $\vec{r}$ again from scratch).

(c) Speed.

$$|\vec{v}(0)| = \sqrt{1^2+(-2)^2+2^2} = \sqrt{1+4+4} = \sqrt{9} = 3 \text{ units/s}$$

(d) Direction of motion.

$$\hat{v}(0) = \frac{\hat{i}-2\hat{j}+2\hat{k}}{3} = \frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$$

(e) $\vec{v}\cdot\vec{a} = (1)(2)+(-2)(0)+(2)(0) = 2 > 0$.

Interpretation: At $t=0$ the particle moves at 3 units per second in the direction $\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$. The acceleration $2\hat{i}$ is along $+x$ and $\vec{v}\cdot\vec{a} = 2 > 0$, so the particle is speeding up slightly while the x-motion is oscillatory/exponential in nature.

Q6 **A cannonball is fired from the ground at an angle 30° with an initial velocity of 100 m/s. Assuming gravity acts downward ($g = 9.8\ \text{m/s}^2$) and neglecting air resistance.**
(a) Formulate the position vector $\vec{r}(t)$ of the cannonball as a function of time. (b) Find the velocity vector at $t = 5$ s. (c) Determine the time of flight (when the cannonball hits the ground).

ہدایت: (a) projectile کا formula لکھیں اور $h=0,\ v_0=100,\ \theta=$ $\vec{r} = (v_0\cos\theta)t\,\hat{i}+\left(h+(v_0\sin\theta)t-\frac{1}{2}gt^2\right)\hat{j}$. (b) $\vec{r}$ کا derivative لے کر $t=5$ رکھیں۔ $30^\circ,\ g=9.8$ رکھیں۔ (c) زمین پر پہنچنے کا مطلب $y(t)=0$؛ اس سے t نکالیں۔

(a) Position vector. Projectile formula:

$$\vec{r}(t) = (v_0\cos\theta)t\,\hat{i}+\left(h+(v_0\sin\theta)t-\tfrac{1}{2}gt^2\right)\hat{j}$$

Fired from the ground, so $h=0$, $v_0=100$, $\theta=30^\circ$, $g=9.8$:

$$\vec{r}(t) = (100\cos 30^\circ)t\,\hat{i}+\left(0+(100\sin 30^\circ)t-\tfrac{1}{2}(9.8)t^2\right)\hat{j}$$
$$\vec{r}(t) = 100\left(\frac{\sqrt{3}}{2}\right)t\,\hat{i}+\left(100\cdot\frac{1}{2}t-4.9t^2\right)\hat{j}$$

$$\vec{r}(t) = 50\sqrt{3}\,t\,\hat{i}+(50t-4.9t^2)\,\hat{j} \qquad \cdots(1)$$

CONCEPT

Projectile motion: horizontal part $x = (v_0\cos\theta)t$ (constant speed); vertical part $y = h+(v_0\sin\theta)t-\frac{1}{2}gt^2$ (gravity). $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \frac{1}{2}$.

(b) Velocity.

$$\vec{v}(t) = \frac{d}{dt}\vec{r} = \frac{d}{dt}(50\sqrt{3}t)\hat{i} + \frac{d}{dt}(50t - 4.9t^2)\hat{j} = 50\sqrt{3}\,\hat{i} + (50 - 9.8t)\,\hat{j}$$

At $t = 5$ s:

$$\vec{v}(5) = 50\sqrt{3}\,\hat{i} + (50 - 9.8(5))\hat{j} = 50\sqrt{3}\,\hat{i} + (50 - 49)\hat{j}$$

$$\boxed{\vec{v}(5) = 50\sqrt{3}\,\hat{i} + \hat{j} \approx 86.6\,\hat{i} + 1\,\hat{j}\ \text{m/s}}$$

(c) Time of flight. The ball hits the ground when $y(t) = 0$. From (1):

$$50t - 4.9t^2 = 0 \;\Rightarrow\; t(50 - 4.9t) = 0$$

$$t = 0 \ \text{(time of launch)} \qquad \text{or} \qquad 50 - 4.9t = 0 \Rightarrow t = \frac{50}{4.9}$$

$$\boxed{t \approx 10.20 \text{ seconds (flight time)}}$$

CONCEPT

"Hits the ground" $\Rightarrow\ y = 0$. The root $t = 0$ is the launch; take the other root.

Correction: the original scan shows $t \approx 10.29$ s, but $50 \div 4.9 = 10.204\ldots$, so the correct value is ≈ 10.20 s.

Q7 **A comet orbits the Sun in an elliptical path such that its position (in AU) at time t (in years) is described by $x = 2\cos t,\ y = \sin t$.**
(a) Write the position vector $\vec{r}(t)$. (b) Find the velocity and speed at $t = \frac{\pi}{2}$. (c) Show that the acceleration vector always points toward the origin.

ہدایت: (a) x اور y کو $\hat{i}, \hat{j}$ کے ساتھ جوڑیں۔ (b) $\vec{v} = \vec{r}'$ نکال کر $t = \frac{\pi}{2}$ رکھیں اور $|\vec{v}|$ لیں۔ (c) $\vec{a} = \vec{v}'$ نکالیں اور دکھائیں کہ $\vec{a} = -\vec{r}$ (مثبت عدد) $\times\ \vec{r}$ کے برابر ہے، یعنی origin کی طرف۔

(a) Position vector. $x(t) = 2\cos t,\ y(t) = \sin t$, so

$$\boxed{\vec{r}(t) = 2\cos t\,\hat{i} + \sin t\,\hat{j} \qquad \cdots(1)}$$

(b) Velocity and speed.

$$\vec{v}(t) = \frac{d}{dt}(2\cos t\,\hat{i} + \sin t\,\hat{j}) = -2\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$\vec{v}\left(\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2}\,\hat{i} + \cos\frac{\pi}{2}\,\hat{j} = -2(1)\hat{i} + 0\,\hat{j} = -2\,\hat{i}$$

$$\text{Speed} = \left|\vec{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2\ \text{AU/year}$$

CONCEPT

Speed $= |\vec{v}| = \sqrt{v_x^2 + v_y^2}$. In general here $|\vec{v}| = \sqrt{4\sin^2 t + \cos^2 t}$.

(c) Acceleration.

$$\vec{a}(t) = \frac{d}{dt}\vec{v} = \frac{d}{dt}(-2\sin t\,\hat{i} + \cos t\,\hat{j}) = -2\cos t\,\hat{i} - \sin t\,\hat{j}$$

$$\vec{a}(t) = -(2\cos t\,\hat{i} + \sin t\,\hat{j}) \;\Rightarrow\; \boxed{\vec{a}(t) = -\vec{r}(t)} \quad \text{(from (1))}$$

Hence $\vec{a}$ is always a negative multiple of the position vector, so it always points toward the origin. ■

CONCEPT

If $\vec{a} = -k\,\vec{r}$ with $k > 0$, the acceleration points from the particle **toward the origin** (opposite to $\vec{r}$).

Q8 **A robotic arm moves such that: its x-position increases linearly at 2 m/s, its y position follows $y(t) = t^2$, its z-position is given by $z(t) = 1 - e^{-t}$.**

(a) Construct the position vector $\vec{r}(t)$. (b) Find the velocity vector at $t = 1$ s. (c) Compute the acceleration at $t = 1$ s.

ہدایت: (a) "linearly 2 m/s" سے $x = 2t$ بنتا ہے ($s = vt$)؛ تینوں components کو $\hat{i}, \hat{j}, \hat{k}$ کے ساتھ لکھیں. (b) $\vec{v} = \vec{r}'$، پھر $t = 1$. (c) $\vec{a} = \vec{v}'$، پھر $t = 1$.

(a) Position vector. x increases linearly at 2 m/s, so $x(t) = 2t$; also $y(t) = t^2,\ z(t) = 1 - e^{-t}$.

$$\vec{r}(t) = 2t\,\hat{i} + t^2\,\hat{j} + (1 - e^{-t})\,\hat{k}$$

(b) Velocity.

$$\vec{v}(t) = \frac{d}{dt}\vec{r} = 2(1)\hat{i} + 2t\,\hat{j} + \big(0 - e^{-t}(-1)\big)\hat{k} = 2\,\hat{i} + 2t\,\hat{j} + e^{-t}\,\hat{k}$$

$$\vec{v}(1) = 2\,\hat{i} + 2(1)\hat{j} + e^{-1}\hat{k} = 2\hat{i} + 2\hat{j} + \frac{1}{e}\,\hat{k}$$

(c) Acceleration.

$$\vec{a}(t) = \frac{d}{dt}(2\hat{i} + 2t\hat{j} + e^{-t}\hat{k}) = 0\,\hat{i} + 2\,\hat{j} - e^{-t}\,\hat{k}$$

$$\vec{a}(1) = 0\,\hat{i} + 2\,\hat{j} - e^{-1}\hat{k} = 2\,\hat{j} - \frac{1}{e}\,\hat{k}$$

CONCEPT

Distance = speed × time: $x = vt = 2t$.

$\dfrac{d}{dt}e^{-t} = -e^{-t}$ and $\dfrac{d}{dt}(1 - e^{-t}) = e^{-t}$.

Q9 **A particle moves along a circular path, described by the position vector $\vec{r}(t) = R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}$. Prove that its velocity is perpendicular to its position vector.**

ہدایت: پہلے $\vec{v} = \vec{r}'$ نکالیں. دو vectors perpendicular ہوتے ہیں جب ان کا dot product صفر ہو، اس لیے $\vec{v} \cdot \vec{r}$ نکالیں اور دکھائیں کہ جواب 0 ہے.

Velocity.

$$\vec{v}(t) = \frac{d}{dt}(R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}) = -R\omega\sin\omega t\,\hat{i} + R\omega\cos\omega t\,\hat{j}$$

Dot product.

$$\vec{v}\cdot\vec{r} = (-R\omega\sin\omega t)(R\cos\omega t) + (R\omega\cos\omega t)(R\sin\omega t)$$

$$= -R^2\omega\sin\omega t\cos\omega t + R^2\omega\sin\omega t\cos\omega t = 0$$

Conclusion. Since $\vec{v}\cdot\vec{r} = 0$, we get $\vec{v} \perp \vec{r}$. ■

CONCEPT

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$, and $\vec{a}\cdot\vec{b} = 0 \iff \vec{a} \perp \vec{b}$.

Chain rule: $\dfrac{d}{dt}\cos\omega t = -\omega\sin\omega t$.

Q10 **A Ferris wheel of radius 20 m rotates with an angular speed of 0.2 rad/s. (a) Construct its position vector $\vec{r}(t)$. (b) Find the passenger's velocity at $t = 10$ s. (c) Show that the acceleration is centripetal (toward the centre).**

ہدایت: (a) دائرے والی حرکت کا formula $\vec{r} = R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}$ لیں اور $R = 20,\ \omega = 0.2$ رکھیں۔ (b) derivative لے کر $t = 10$ رکھیں۔ (c) $\vec{a} = \vec{v}'$ نکال کر دکھائیں کہ $\vec{a} = (\text{منفی عدد}) \times \vec{r}$ ہے۔

CONCEPT

Circular motion: $\vec{a} = -\omega^2\vec{r}$. Here $\omega^2 = (0.2)^2 = 0.04$. Use **radians** for $\sin 2,\ \cos 2$ ($\sin 2 \approx 0.909,\ \cos 2 \approx -0.416$).

(a) Position vector. For circular motion in a plane $\vec{r}(t) = R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}$. Put $R = 20,\ \omega = 0.2$:

$$\vec{r}(t) = 20\cos(0.2t)\,\hat{i} + 20\sin(0.2t)\,\hat{j} \qquad \cdots(1)$$

(b) Velocity.

$$\vec{v}(t) = \frac{d}{dt}\vec{r} = -20\sin(0.2t)(0.2)\hat{i} + 20\cos(0.2t)(0.2)\hat{j}$$
$$= -4\sin(0.2t)\,\hat{i} + 4\cos(0.2t)\,\hat{j}$$

At $t = 10$ s ($0.2 \times 10 = 2$ rad):

$$\vec{v}(10) = -4\sin 2\,\hat{i} + 4\cos 2\,\hat{j} \approx -3.64\,\hat{i} - 1.66\,\hat{j}\ \text{m/s}$$

(c) Acceleration.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -4\cos(0.2t)(0.2)\hat{i} - 4\sin(0.2t)(0.2)\hat{j} = -0.8\cos(0.2t)\,\hat{i} - 0.8\sin(0.2t)\,\hat{j}$$

$$\vec{a}(t) = -0.04\big[20\cos(0.2t)\,\hat{i} + 20\sin(0.2t)\,\hat{j}\big] \ \Rightarrow\ \boxed{\vec{a}(t) = -0.04\,\vec{r}(t)} \quad \text{(from (1))}$$

The acceleration is a negative multiple of $\vec{r}$, so it points toward the centre (centripetal). ■

Correction: the scan writes the $\hat{i}$-coefficient of $\vec{a}$ as -0.08; it is $-4(0.2) = -0.8$, and then $\vec{a} = -0.04\,\vec{r}$ is consistent.

Q11 **A quarterback throws a football with a position vector $\vec{r}(t) = 20t\,\hat{i} + (20t - 4.9t^2)\,\hat{j}$.**
(a) Find the time of flight before it hits the ground. (b) At which time did the football reach its maximum height? (c) What is the maximum height reached?

ہدایت: (a) زمین پر پہنچنے پر $y(t) = 0$ رکھیں۔ (b) سب سے اونچے مقام پر عمودی رفتار $v_y = \frac{dy}{dt} = 0$ ہوتی ہے، اس سے t نکالیں۔ (c) اس t کو $y(t)$ میں رکھ کر زیادہ سے زیادہ اونچائی نکالیں۔

(a) Time of flight. It hits the ground when $y(t) = 0$:

$$20t - 4.9t^2 = 0 \;\Rightarrow\; t(20 - 4.9t) = 0$$

$$t = 0 \text{ (kick-off)} \quad \text{or} \quad 4.9t = 20 \Rightarrow t = \frac{20}{4.9}$$

$$t \approx 4.08 \text{ seconds (flight time)}$$

CONCEPT

Max height $\Rightarrow$ vertical velocity $v_y = 0$ (the ball stops rising before it falls). It is exactly half of the flight time: $2.04 \approx \frac{4.08}{2}$.

(b) Time of maximum height. At the top, $v_y = 0$:

$$\frac{d}{dt}y(t) = \frac{d}{dt}(20t - 4.9t^2) = 20 - 9.8t = 0 \;\Rightarrow\; t = \frac{20}{9.8}$$

$$t \approx 2.04 \text{ seconds}$$

(c) Maximum height.

$$y(2.04) = 20(2.04) - 4.9(2.04)^2 = 40.8 - 4.9(4.1616) = 40.8 - 20.39$$

$$\text{Max. height} \approx 20.41 \text{ metres}$$

Note: with exact $t = \frac{20}{9.8} = 2.0408$: $y = 40.82 - 20.41 = 20.41$ m. Using the rounded $t = 2.04$ gives $40.8 - 20.39 = 20.41$ m (the scan wrote the middle step as "4.8 − 20.39", a typing slip).